EFFECTS OF ANGULAR VIBRATION ON THE PERFORMANCE OF A SINGLE - DEGREE - OF - FREEDOM INTEGRATING GYROSCOPE

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ABSTRACT

Recent studies have shown that various inaccuracies occur when a single-degree-of-freedom gyroscope is subjected to an angular vibratory environment. Under such conditions the performance of the gyroscope has been analyzed and the nature of the inaccuracies investigated. Errors due to "coning", "dynamic unbalance", and "sculling" have been included in the investigation.

Values of the inaccuracies representative of those expected from tests using the precision angular vibrator developed by the Instrumentation Laboratory of the Massachusetts Institute of Technology are presented.

By reference to a dynamic model the equation of motion was obtained. A suitable solution to the equation was determined in series form for the case of sinusoidal inputs simultaneously applied about two axes. No restrictions are placed on the frequency of the applied motions. The terms in the series solution reveal the nature of the angular displacement of the float with respect to inertial space about the output axis for two cases: (1) zero elastic restraint, (2) finite elastic restraint.

It is strongly recommended that correlation of actual experimental results with the developed theory be obtained prior to further theoretical considerations.

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LIST OF ILLUSTRATIONS

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OBJECT

The object of this thesis is to analyze the performance of a floated single-degree-of-freedom integrating gyroscope subjected to angular vibrations in order to determine the inaccuracies that may result from such an environment.

CHAPTER 1

INTRODUCTION

The age of the golden sixties is upon us. Ushered in with this decade is an era of accomplishment which has raised the importance of the gyroscope to an all-time high. This era of great advancement includes those fields of Science and Engineering which are brought to the fore when one thinks of space. Because of space, these fields have been searched diligently for methods of navigation and control and from this searching the gyroscope has emerged, holding a position of extreme prominence.

With this relatively new stature, the gyroscope has naturally been investigated in efforts to improve its performance as well as to predict what may be expected in the future. The floated single-degree-of-freedom integrating gyroscope, an important element in most space navigation and guidance systems, is the subject of this analysis. The SDFI gyroscope has been interrogated from the standpoint of being subjected to varying vibratory inputs which may introduce error. These vibrations may be of very low frequency due to the bending modes of long missile bodies or they may be of high frequencies associated with aerodynamic flutters and buzzing. Through the theoretical results, it is hoped that undesirable effects

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caused by such vibratory motions will be recognized and later minimized when actual verifying data is available.

The floated single-degree-of-freedom integrating gyroscope is an instrument capable of sensing angular velocities to a high degree of accuracy. When utilized in inertial guidance systems, the gyroscope must have considerable stability and extreme sensitivity. The vibratory environment in which the gyro functions may introduce unwanted angular outputs. This degradation of desired performance is the result of both linear and angular oscillations. An analysis of the undesired effects is made in order to determine the nature and magnitudes of the errors involved.

This investigation was conducted with the view that the Massachusetts Institute of Technology Instrumentation Laboratory will conduct tests on a precision angular vibrator. A previous investigation has disclosed what may be expected if two equal frequencies are simultaneously applied to the output and spin reference axes of a typical single-degree-of-freedom gyroscope. The present analysis is concerned with any two frequencies simultaneously applied to the output and spin reference axes of a SDFI gyroscope.

The existing equation of motion is based on a simplified dynamic model which accounts for damping, compliance, and inertia. As no closed form solution to the equation is known, a series form was used which gives a very clear representation of the gyroscopic drift terms. The series solution appears easy to handle and analyze in view of the theory of superposition. The terms seem reasonable in magnitude and should be easily verified by the Massachusetts Institute of Technology angular vibrator.

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CHAPTER 2

THE FLOATED SINGLE-DEGREE-OF-FREEDOM INTEGRATING GYROSCOPE

2.1 Design Features

The types and locations of components within the ideal floated single-degree-of-freedom integrating gyroscope unit is pictorially shown in Fig. 2-1. These components are the gyroscopic element, gimbal, viscous damper, microsyn torque generator, microsyn signal generator, gimbal bearings, and the case.

The gyroscopic element and gimbal are hermetically sealed in a cylindrical shell which is separated from the outer case by a high density fluid. The case is also hermetically sealed. The two shells form a thin-walled cylinder that performs two vital functions when completely filled with a high density fluid:

1. it provides a viscous damping torque that opposes the precessional torque generated by the gyro element,

2. it provides a buoyant force that neutralizes the weight of the gyro element and its supporting structure.

The clearance between the gimbal float and the case plus the fluid viscosity determines the damping of the viscous damper; the fluid density and the amount of displaced fluid determine the buoyant force.

Power is conducted to the hysteresis type alternating-current motor by flex leads. The entire float assembly, or torque summing member, is initially aligned with the case by jewel bearings. Final alignment is by magnetic suspension. The suspension (elastic restraint to float radial motions) is provided by the microsyns in conjunction with their primary functions of angle measurement and torque generation. The magnetic forces necessary for suspension originate with the change in excitation currents of properly tuned microsyn coils as a result of the radial displacement of the microsyn rotor from its centered position. Figure 2-2 is a sectioned view of a typical floated single-degree-of-freedom integrating gyroscope. A more detailed discussion of the single-degree-of-freedom integrating gyroscope may be found in Refs. 1 and 2.

2.2 Operation

An ideal single-degree-of-freedom integrating gyroscope receives as inputs, the angular velocity of its case with respect to inertial space about the input axis and electrical current input signals to the torque generator. The unit generates an output voltage proportional to the inputs. In open loop operation, the gyro output signal would represent the angular deviation of the gyro case from a reference orientation. The reference is established from initial conditions and by command signals to the torque generator.

In operation, the gyro unit is housed in a thermally controlled environment that maintains the temperature to within a few degrees Fahrenheit. This control is important because temperature affects the density and viscosity of the fluid.

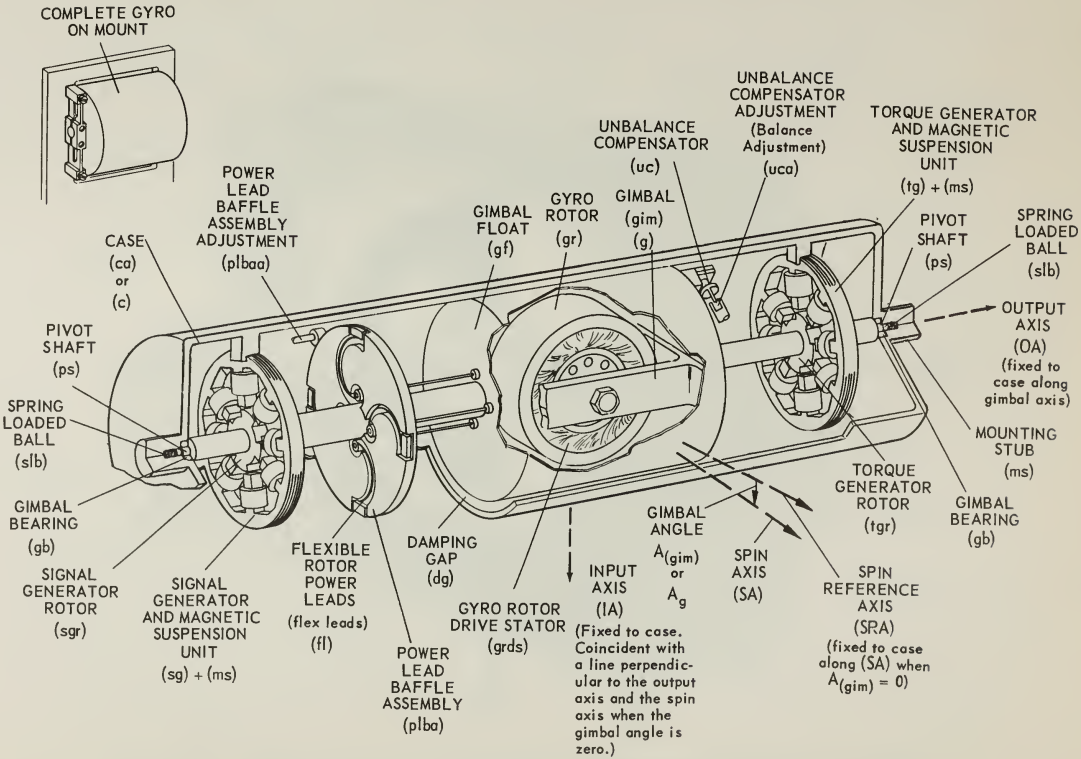


Fig. 2-1. Pictorial schematic of the single-degree-of-freedom floated integrating gyro unit.

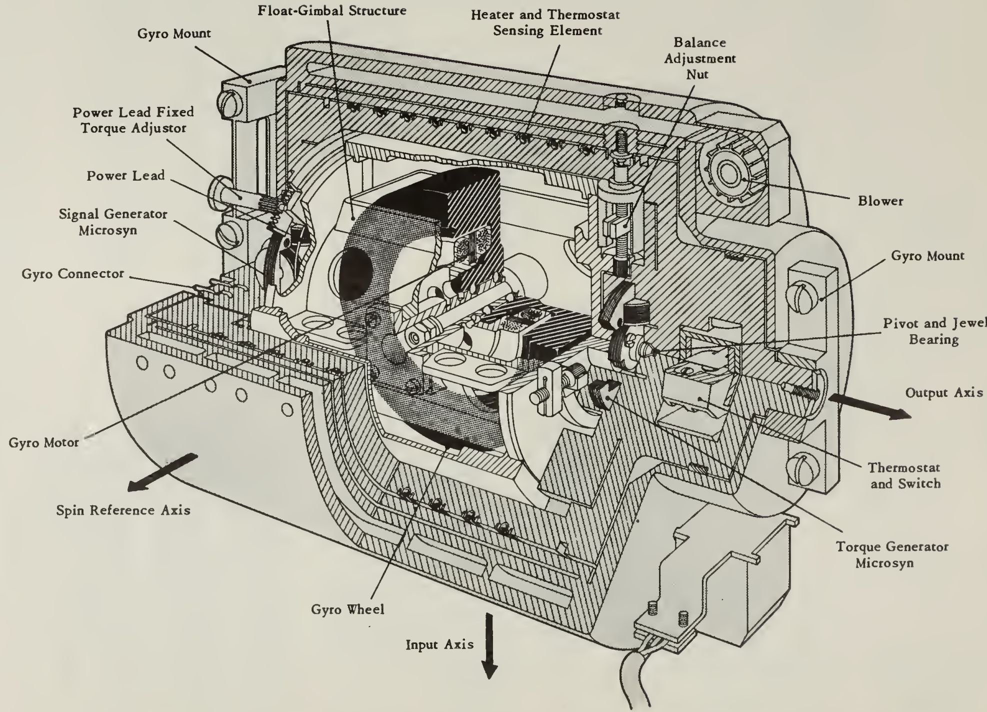
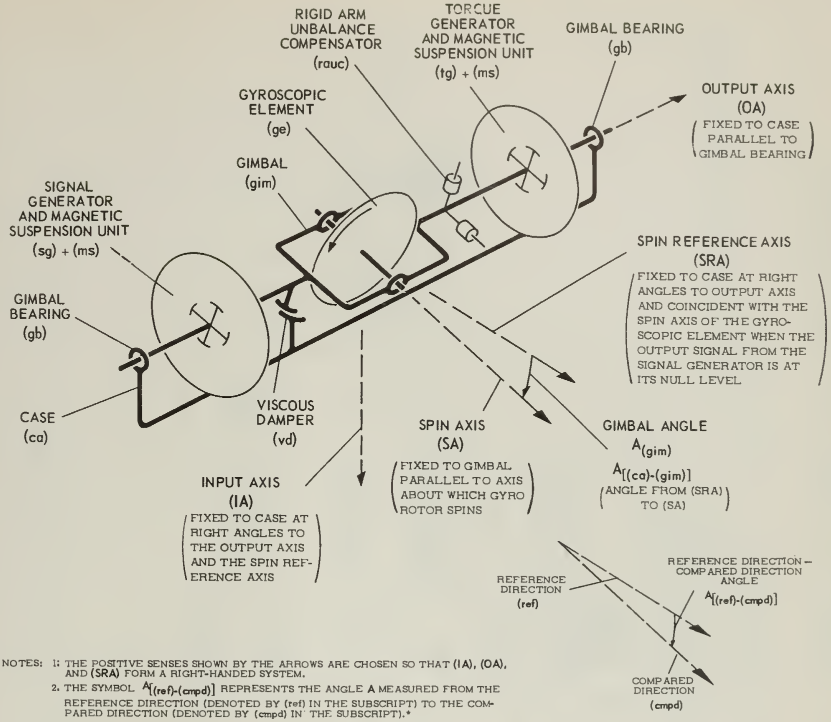


Fig. 2-2 Sectioned view of a typical floated single-degree-of-freedom integrating gyre.



NOTES: 1. THE POSITIVE SENSES SHOWN BY THE ARROWS ARE CHOSEN SO THAT (IA), (OA), AND (SRA) FORM A RIGHT-HANDED SYSTEM.

2. THE SYMBOL $A_{[(ref)-(cmpd)]}$ REPRESENTS THE ANGLE A MEASURED FROM THE REFERENCE DIRECTION (DENOTED BY (ref) IN THE SUBSCRIPT) TO THE COMPARED DIRECTION (DENOTED BY (cmpd) IN THE SUBSCRIPT).*

DEFINITIONS:

CASE, (ca) or (c)
THE STRUCTURE THAT GIVES SUPPORT FOR THE INTERNAL WORKING PARTS OF THE GYRO UNIT, ENCLOSES THESE PARTS, AND CARRIES PROVISIONS FOR EXTERNAL CONNECTIONS OF ALL KINDS.

TORQUE GENERATOR AND MAGNETIC SUSPENSION UNIT, (tg) + (ms)
COMPONENT FOR (a) RECEIVING INPUT SIGNALS AND PRODUCING A CORRESPONDING OUTPUT TORQUE APPLIED TO THE GIMBAL ABOUT THE OUTPUT AXIS AND (b) ALSO PROVIDING MAGNETIC FORCE ACTING TO CENTRALIZE THE PIVOT SHAFT IN THE BEARING.

DAMPER, (dmp) or (d)
SUBSYSTEM RECEIVING THE ANGULAR VELOCITY OF THE GIMBAL WITH RESPECT TO THE CASE AS ITS INPUT AND PRODUCING AS THE OUTPUT A RETARDING TORQUE ACTING ON THE GIMBAL ABOUT THE OUTPUT AXIS WITH A MAGNITUDE PROPORTIONAL TO THE MAGNITUDE OF ITS ANGULAR VELOCITY INPUT.

GYRO UNIT, (gu)
THE ENTITY MADE UP OF THE COMPONENTS REPRESENTED IN THIS DIAGRAM AND ALL THE ADDITIONAL PARTS NECESSARY FOR A SINGLE PACKAGE TO CARRY OUT THE FUNCTIONS OF A GYRO UNIT.

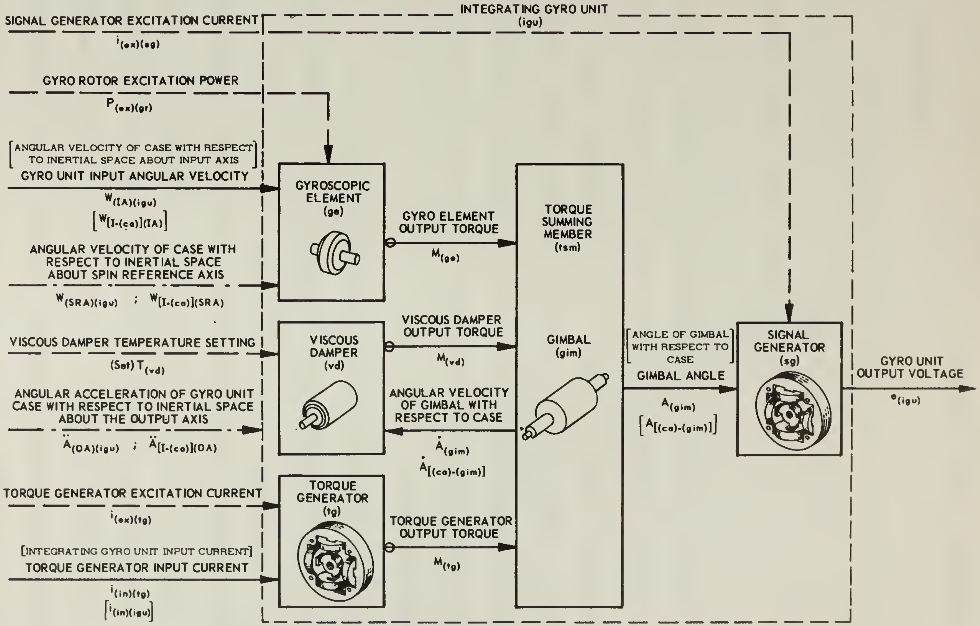
SIGNAL GENERATOR AND MAGNETIC SUSPENSION UNIT, (sg) + (ms)
COMPONENT FOR (a) RECEIVING THE ANGLE OF THE SPIN AXIS WITH RESPECT TO THE CASE AS ITS INPUT AND PRODUCING A CORRESPONDING SIGNAL THAT SERVES AS THE OUTPUT SIGNAL FROM THE GYRO UNIT AND (b) ALSO PROVIDING MAGNETIC FORCE ACTING TO CENTRALIZE THE PIVOT SHAFT IN THE BEARING.

GIMBAL, (gim)
STRUCTURE CARRYING THE BEARINGS FOR THE SPINNING ROTOR OF THE GYROSCOPIC ELEMENT, ROTORS FOR THE TORQUE GENERATOR AND SIGNAL GENERATOR, PART OF THE DAMPER, FLOAT SEALS AND STRUCTURE, BALANCE ADJUSTMENTS, STOPS, PIVOTS, ETC.

RIGID ARM UNBALANCE COMPENSATOR, (rauc)
SCREW ADJUSTMENTS FOR SHIFTING THE CENTER OF MASS OF THE GIMBAL FLOAT TO A POSITION SUBSTANTIALLY ON THE OUTPUT AXIS.

* A DISCUSSION OF GENERALIZED CONVENTIONS FOR SELF-DEFINING SYMBOLS OF WHICH $A_{[(ref)-(cmpd)]}$ IS AN EXAMPLE IS GIVEN BY DRAPER, McKAY AND LEES IN INSTRUMENT ENGINEERING (10), VOL. I.

Fig. 2-3 Line schematic diagram for the single-degree-of-freedom integrating gyro unit.



NOTES: 1. INERTIA REACTION EFFECTS APPEAR WHENEVER MATTER IS ACCELERATED. THIS MAKES IT IMPOSSIBLE TO ISOLATE ANY SINGLE OPERATING COMPONENT AS THE SEAT OF INERTIA REACTION TORQUE AND NO SUCH COMPONENT IS REPRESENTED IN THE DIAGRAM. IT IS UNDERSTOOD THAT INERTIA REACTION TORQUE EXISTS AND A TERM IS INCLUDED IN THE PERFORMANCE EQUATION BASED ON THE RESULTANT EFFECT ASSOCIATED WITH THE ACCELERATED PARTS.

2. THE GYRO UNIT TEMPERATURE CONTROL POWER IS SUPPLIED TO A MOUNTING BLOCK ADAPTED TO RECEIVE THE GYRO UNIT CASE. THE FLOW OF POWER IS CONTROLLED BY THE DAMPER TEMPERATURE SETTING.

ARROW SHOWS THE ESSENTIAL DIRECTION OF TRANSFER FOR THE QUANTITY REPRESENTED. THE CIRCLE SHOWS THE TRANSFER IS REVERSIBLE, THAT IS, A REACTION TORQUE ALWAYS ACCOMPANIES AN APPLIED TORQUE

ACTUATING INPUT

MODIFYING INPUT

INTERFERING INPUT

Fig. 2-4 Functional diagram for the floated single-degree-of-freedom integrating gyro unit.

The torque and signal generators are microsyn units. The rotors of these units are attached to the gimbal float while the stators are fixed to the case.

A line schematic diagram is shown in Fig. 2-3 and a functional diagram in Fig. 2-4. If the gyroscopic element output is the only torque applied to the torque-summing member of the gyro unit, it is balanced by the viscous shear torque. The viscous torque is proportional to the angular velocity of the floated gimbal assembly relative to the case. This torque balance causes the output angular velocity to be proportional to the input angular velocity. Thus, the angle of the floated gimbal with respect to the case is proportional to the angular displacement of the case about the input axis, measured with respect to inertial space. In other words, the angle through which the float turns with respect to the case represents the time integral of the gyro input angular velocity.

The torque generator permits introduction of command orientation signals. The torque generator applies a torque to the float, proportional to the current supplied the torque generator. If there is no gyroscopic torque, the output of the torque generator causes the float to rotate (against the viscous shear torque of the damper) at a rate proportional to the current input to the torque generator. The gyro output is therefore equivalent to an angular displacement about the gyro input axis. The output axis angular displacement is measured by the microsyn signal generator. The output voltage of the signal generator is utilized for indication or control.

For the gyroscope to be a linear instrument, the angle through which the float turns with respect to the case must be kept small. For

[illegible]

the integrating gyro this implies closed loop operation. There are two types of closed (servo) loops presently in common use:

1. a rate feedback loop in which the signal generator output voltage is amplified and applied to the gyro torque generator to act as an elastic restraint,
2. a stabilization servo loop in which the signal generator output voltage is amplified and applied to a servo motor that supplies an angular rate about the gyro input axis.

CHAPTER 3

INACCURACIES DUE TO SIMULTANEOUS VIBRATIONS ABOUT THE CASE INPUT AND SPIN AXES

3.1 General

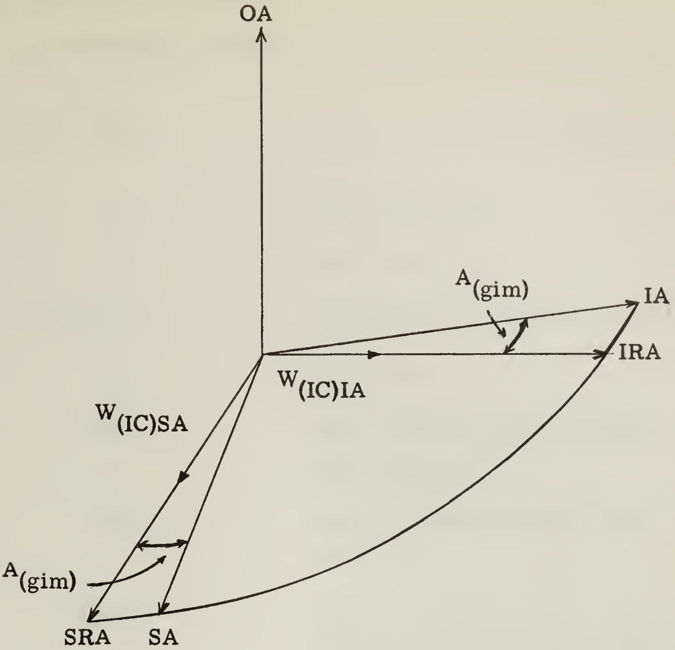
This chapter deals with the response of the single-degree-of-freedom integrating gyroscope when simultaneous angular vibrations are applied about the case input and spin axes. The resulting error torques are determined and their physical origin analyzed.

It is assumed that when an angular vibration is present in the plane of the gyro case input and spin axes, components exist about both axes. The torque applied externally to the gyroscope is developed using basic gyro theory.

3.2 Coordinate System and Symbols

3.2.1 Coordinate System

The coordinate systems and the relationship of the angular velocity components are shown in Fig. 3-1. The spin reference axis (SRA), input reference axis (IRA) and output reference axis (ORA) are orthogonal axes fixed to the gyro case. The gyroscopic element is considered to be rigidly fixed to the float so that the float input axis (IA), spin axis (SA) and output axis (OA) are also orthogonal.



$$W_{(IF)SA} = W_{(IC)SA} \cos A_{(gim)} + W_{(IC)IA} \sin A_{(gim)}$$

$$W_{(IF)IA} = W_{(IC)IA} \cos A_{(gim)} - W_{(IC)SA} \sin A_{(gim)}$$

Fig. 3-1. Gyroscope Coordinate System and Angular Velocity Components

3.2.2 Definition of Symbols

The following symbols and definitions will be used in the analysis.

$A_{(C-F)OA} = A_{(gim)}$	Angle of float with respect to case
IA	Float input axis
IRA	Case input axis
ORA = OA	Output axis of both the case and float
SA	Spin axis
SRA	Spin reference axis (coincident with SA when $A_{(gim)} = 0$)
W_{SP}	Angular velocity of gyro wheel about spin axis
$W_{(IC)IA}$	Angular velocity of case with respect to inertial space about case input axis
$W_{(IF)IA}$	Angular velocity of float with respect to inertial space about float input axis.
$W_{(IC)SA}$	Angular velocity of case with respect to inertial space about case spin axis
$W_{(IF)SA}$	Angular velocity of float with respect to inertial space about float spin axis
I_{IA}	Moment of inertia of wheel and float about float input axis
I_{SA}	Moment of inertia of wheel and float about float spin axis

I_W	Moment of inertia of gyro wheel about float spin axis
H	Total angular momentum
H_{NS}	Nonspin angular momentum
H_{SP}	Angular momentum of spinning wheel
$M_{(app)}$	Torque applied to the gyro unit input axis
$\left.\dfrac{d\vec{H}}{dt}\right\|_I$	Rate of change of angular momentum with respect to inertial space
$\left.\dfrac{dH}{dt}\right\|_F$	Rate of change of angular momentum in float coordinates
$M_{(out)}$	Torque output about gyro output axis
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors along orthogonal axes forming right-hand set
n	Linear acceleration
n_x	Linear acceleration component along platform axis having $\bar{i}$ as unit vector
n_y	Linear acceleration component along platform axis having $\bar{j}$ as unit vector
n_z	Linear acceleration component along platform axis having $\bar{k}$ as unit vector
$n_{x_o}, n_{y_o}, n_{z_o}$	Linear acceleration components along non-rotating axes having $\bar{i}, \bar{j}, \bar{k}$ as unit vectors
ψ, θ, ϕ	Euler angles

3.3 Gyro Response to Angular Vibrations

3.3.1 Applied Torque due to Angular Vibrations

In vector notation the torque applied[3] to the gyro is

$$\overline{M}_{(app)} = \left.\frac{d\overline{H}}{dt}\right|_I = \left.\frac{d\overline{H}}{dt}\right|_F + \overline{W}_{(IF)} \times \overline{H} \qquad (3\text{-}1)$$

By Newton's Law of Action and Reaction, the gyro unit output torque is

$$\overline{M}_{(out)} = -\overline{M}_{(app)} = \overline{H} \times \overline{W}_{(IF)} - \left.\frac{d\overline{H}}{dt}\right|_F \qquad (3\text{-}2)$$

Using the relationships shown in Fig. 3-1, the following vectors may be written:

$$\overline{W}_{(IF)} = \overline{i}\, W_{(IF)SA} + \overline{j}\, W_{(IF)IA}$$

$$\overline{H} = \overline{H}_{NS} + \overline{H}_{SP} = \overline{i}(I_{SA} W_{(IF)SA} + I_W W_{SP}) + \overline{j}(I_{IA} W_{(IF)IA})$$

$$\overline{W}_{(IF)} \times \overline{H} = \overline{k}[(I_{IA} - I_{SA})\, W_{(IF)SA} W_{(IF)IA} + H_{SP} W_{(IF)IA}] \qquad (3\text{-}3)$$

The relationships from Fig. 3-1 are then substituted into Eq. (3-3) to obtain Eq. (3-4).

$$\begin{aligned}\overline{M}_{(app)} = \overline{k}[&(I_{IA} - I_{SA})(\cos^2 A_{(gim)} - \sin^2 A_{(gim)})(W_{(IC)SA} W_{(IC)IA}) \\ &+ (I_{IA} - I_{SA})(W^2_{(IC)IA} - W^2_{(IC)SA}) \sin A_{(gim)} \cos A_{(gim)} \\ &- H_{SP}(W_{(IC)IA} \cos A_{(gim)} - W_{(IC)SA} \sin A_{(gim)})] \qquad (3\text{-}4)\end{aligned}$$

For a practical single-degree-of-freedom gyroscope the gimbal angle, $A_{(gim)}$ usually does not exceed one degree.[4] Therefore, the equation may be simplified by making use of small angle approximations and neglecting second order terms in $A_{(gim)}$. The torque equation is reduced to:

$$\overline{M}_{(app)} = \overline{k}[(I_{IA} - I_{SA}) W_{(IC)SA} W_{(IC)IA} + (I_{IA} - I_{SA}) A_{(gim)} \times$$

$$(W^2_{(IC)IA} - W^2_{(IC)SA}) - H_{SP}(W_{(IC)IA} - A_{(gim)} W_{(IC)SA})] \tag{3-5}$$

For a single-degree-of-freedom gyroscope only torques about the output axis are considered. The rate of change of angular momentum in float coordinates does not contribute an output axis torque. Equation (3-5) represents the applied torque of the gyro and, since it is all along a single axis, vector notation will not be necessary.

$$M_{(app)} = [(I_{IA} - I_{SA}) W_{(IC)SA} W_{(IC)IA} + (I_{IA} - I_{SA}) A_{(gim)}(W^2_{(IC)IA} - W^2_{(IC)SA})$$

$$- H_{SP} (W_{(IC)IA} - A_{(gim)} (W_{(IC)SA})] \tag{3-5a}$$

The terms of Eq. (3-5) represent error or undesired torques. A single-degree-of-freedom gyroscope, operating in a closed loop, would have an angular displacement from the null or zero position that would be in error due to these undesired torques. An examination of these error torques will now be made.

3.3.2 Dynamic Torques

The first torque component to be considered will be

$$(I_{IA} - I_{SA}) W_{(IC)SA} W_{(IC)IA} \tag{3-6}$$

It is evident that the nature of the error will depend upon the variation of the two simultaneous angular velocities and the float inertias about the spin and input axes. This error is known as <u>anisoinertia</u>[5] or dynamic unbalance. It could be eliminated by building gyro units having equal inertias about both axes.

[illegible]

Next consider the second term of Eq. (3-5)

$$(I_{IA} - I_{SA}) A_{(gim)} (W^2_{(IC)IA} - W^2_{(IC)SA}) \qquad (3\text{-}7)$$

In the above expression the nature of the error will be determined by the difference of the inertias as well as the difference of squared angular velocities. Variation of gimbal angle must also be considered since it is not a constant.

Expressions (3-6) and (3-7) are dynamic torques[6] and may be interpreted as representing four angular momentums and their associated precessional angular velocities. Consider that there are four angular velocity components and four corresponding angular momentums.

Momentum Components	Angular Velocity Components
$I_{SA} W_{(IC)SA}$	$W_{(IC)SA}$
$-A_{(gim)} I_{IA} W_{(IC)SA}$	$-A_{(gim)} W_{(IC)SA}$
$I_{IA} W_{(IC)IA}$	$W_{(IC)IA}$
$A_{(gim)} I_{SA} W_{(IC)IA}$	$A_{(gim)} W_{(IC)IA}$

The total "dynamical torque"[6] when each forced precession is considered is:

$$\begin{aligned} M_D = {} & -I_{SA} W_{(IC)SA}(W_{(IC)IA} - A_{(gim)} W_{(IC)SA}) \\ & - I_{IA} A_{(gim)} W_{(IC)SA} (W_{(IC)SA} + A_{(gim)} W_{(IC)IA}) \\ & + I_{IA} W_{(IC)IA} (W_{(IC)SA} + A_{(gim)} W_{(IC)IA}) \\ & - I_{SA} A_{(gim)} W_{(IC)IA} (W_{(IC)IA} - A_{(gim)} W_{(IC)SA}) \end{aligned}$$

The part in parentheses for each of the above terms is the forced precessional rate while the part before the parentheses is the angular momentum. For example, the angular momentum component, $I_{SA}W_{(IC)SA}$ precesses about the input axis with the angular velocity, $W_{(IC)IA} - A_{(gim)}W_{(IC)SA}$. When the equation is expanded and second order terms of $A_{(gim)}$ neglected, the expression for dynamic torque is obtained.

$$M_D = (I_{IA} - I_{SA})\, W_{(IC)IA} W_{(IC)SA} + (I_{IA} - I_{SA})\, A_{(gim)} (W^2_{(IC)IA} - W^2_{(IC)SA}) \tag{3-8}$$

3. 3. 3 Gyroscopic Torque

The applied gyroscopic torque on the gyro is

$$M_{ge} = -H_{SP}(W_{(IC)IA} - A_{(gim)}\, W_{(IC)SA}) = -H_{SP}\, W_{(IF)IA} \tag{3-9}$$

Variation of this torque will, as in previous considerations, be dependent on the angular velocities and the gimbal angle. The component, $H_{SP}\, A_{(gim)}\, W_{(IC)SA}$, is of special significance and is called Coning Error. (7)

3. 4 Coning

3. 4. 1 General Discussion of Coning

The coning error defined above is more usually considered a cross coupling error. It is a result of oscillations in the plane of the case input and spin axes. Comparing the form of the coning expression from the next section with that of cross coupling reveals a close similarity. Because of this similarity, it is possible to consider cross coupling

as another cause of the coning error. This form of coning error may be reduced by restraining rotations about the output axis. A tight servo loop would keep the error negligible.

If the gyro input axis has a motion such that it returns periodically to its original position, then the input axis will trace out a path in space. This motion is known as the coning motion and is present even in a theoretically perfect instrument.

The coning error is a geometric effect that allows the gyro to respond to an input rate produced by the coning motion.(7) Almost any motion of the case creates coning. Consider oscillations in the plane of the case input and spin axes. The component along the input axis will cause the gyro to precess about the output axis. When the oscillating motion about the spin reference axis is combined with this precession, the result is a coning motion of the float input axis. As another example, if there is oscillation of the case in the spin axis - output axis plane, coning would again result because the damping fluid will drag the float around.

When single-degree-of-freedom gyros are used for stabilization, a coning error exists that cannot be eliminated.

3.4.2 Platform Coning Effect(8)

Consider two coordinate systems:

$\bar{i}_p, \bar{j}_p, \bar{k}_p$	platform coordinate system
$\bar{i}_o, \bar{j}_o, \bar{k}_o$	coordinate system defining the orientation the platform servos are trying to maintain

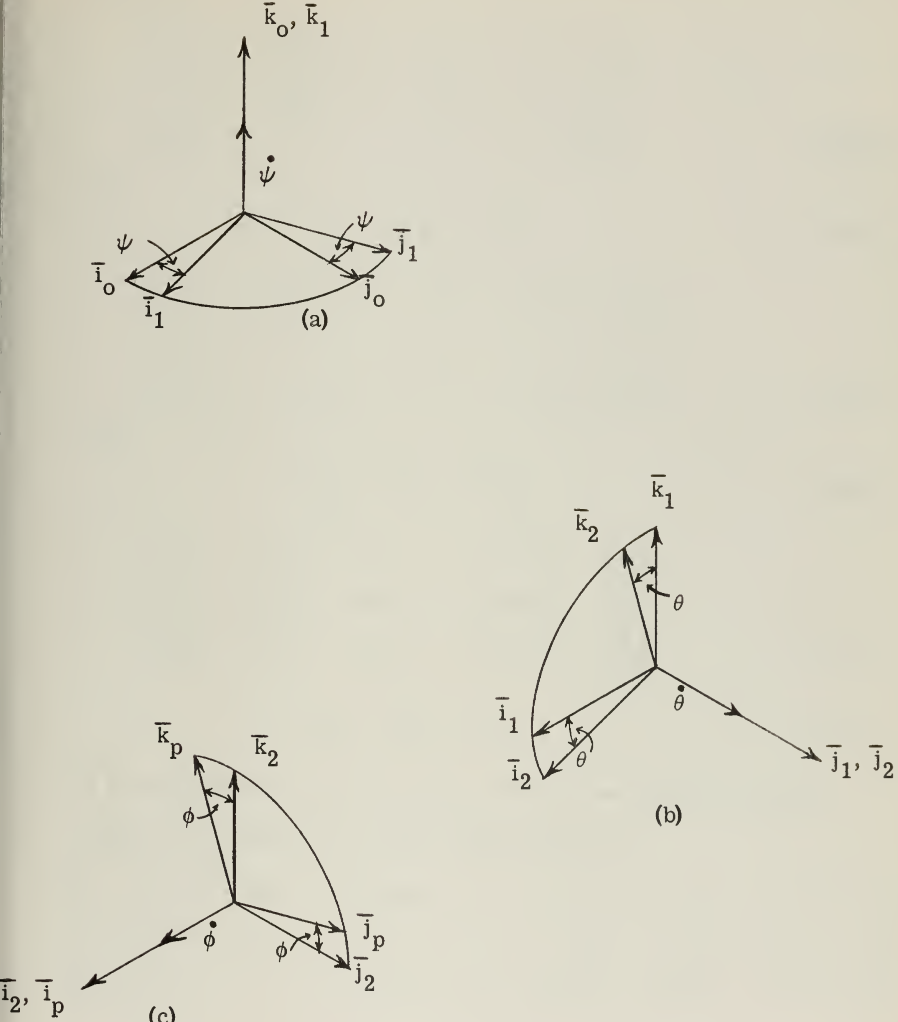


Fig. 3-2. Euler Angle Relations Between Platform and Desired Orientation Coordinates

By use of the Euler angle rotations, taken in the order shown in Fig. 3-2, the coordinate transformations, using matrix notation, are as follows:

$$\begin{bmatrix} \bar{i}_o \\ \bar{j}_o \\ \bar{k}_o \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{bmatrix} \tag{3-10}$$

and

$$\bar{\omega}_{10} = \dot{\psi}\,\bar{k}_o = \dot{\psi}\,\bar{k}_1 \tag{3-11}$$

$$\begin{bmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{bmatrix} \tag{3-12}$$

and

$$\bar{\omega}_{21} = \dot{\theta}\,\bar{j}_1 = \dot{\theta}\,\bar{j}_2 \tag{3-13}$$

$$\begin{bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \bar{i}_p \\ \bar{j}_p \\ \bar{k}_p \end{bmatrix} \tag{3-14}$$

$$\bar{\omega}_{p2} = \dot{\phi}\,\bar{i}_2 = \dot{\phi}\,\bar{i}_p \tag{3-15}$$

It is possible to combine the three matrices and simplify them by making the standard small angle approximations:

$$\begin{bmatrix} \bar{i}_o \\ \bar{j}_o \\ \bar{k}_o \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \bar{i}_p \\ \bar{j}_p \\ \bar{k}_p \end{bmatrix}$$

$$\begin{bmatrix} \bar{i}_o \\ \bar{j}_o \\ \bar{k}_o \end{bmatrix} = \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \begin{bmatrix} \bar{i}_p \\ \bar{j}_p \\ \bar{k}_p \end{bmatrix} \tag{3-16}$$

The angular velocity of the platform coordinate system with respect to the orientation coordinate system may be defined as:

$$\bar{\omega}_{po} = p\,\bar{i}_p + q\,\bar{j}_p + r\,\bar{k}_p \tag{3-17}$$

From Eqs. (3-10), (3-11), and (3-12) we also have

$$\begin{aligned} \bar{\omega}_{po} &= \bar{\omega}_{p2} + \bar{\omega}_{21} + \bar{\omega}_{10} \\ &= \dot{\phi}\,\bar{i}_p + \dot{\theta}\,\bar{j}_2 + \dot{\psi}\,\bar{k}_1 \end{aligned} \tag{3-18}$$

In order to express Eq. (3-18) in platform coordinates, substitute relation for $\bar{k}_1$ from Eq. (3-12) and then make use of Eq. (3-14) as follows:

$$\begin{aligned} \bar{\omega}_{po} &= \dot{\phi}\,\bar{i}_p + \dot{\theta}\,\bar{j}_2 + \dot{\psi}(-\sin\theta\,\bar{i}_2 + \cos\theta\,\bar{k}_2) \\ &= \dot{\phi}\,\bar{i}_p + [-\dot{\psi}\sin\theta \quad \dot{\theta} \quad \dot{\psi}\cos\theta] \begin{bmatrix} \bar{i}_2 \\ \bar{j}_2 \\ \bar{k}_2 \end{bmatrix} \\ &= \dot{\phi}\,\bar{i}_p + [-\dot{\psi}\sin\theta \quad \dot{\theta} \quad \dot{\psi}\cos\theta] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \bar{i}_p \\ \bar{j}_p \\ \bar{k}_p \end{bmatrix} \\ &= (\dot{\phi} - \dot{\psi}\sin\theta)\,\bar{i}_p + (\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi)\,\bar{j}_p \\ &\qquad + (-\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi)\,\bar{k}_p \end{aligned}$$

Assuming small angles, a comparison with Eq. (3-17) gives

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[illegible]

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$$
\begin{aligned}
p &= \dot{\phi} - \theta \dot{\psi} \\
q &= \dot{\theta} + \phi \dot{\psi} \\
r &= \dot{\psi} - \phi \dot{\theta}
\end{aligned}
\tag{3-19}
$$

The rates of change of the Eulerian angles are

$$
\begin{aligned}
\dot{\phi} &= p + \theta \dot{\psi} = p + \theta (r + \phi \dot{\theta}) \approx p + r\theta \\
\dot{\theta} &= q - \phi \dot{\psi} = q - \phi (r + \phi \dot{\theta}) \approx q - r\phi \\
\dot{\psi} &= r + \phi \dot{\theta} = r + \phi (q - \phi \dot{\psi}) \approx r + \phi q
\end{aligned}
\tag{3-20}
$$

Integrating Eq. (3-20) to obtain the expressions for the Euler angles we obtain:

$$
\begin{aligned}
\phi(t) &= \int_0^t p\,dt + \int_0^t r\theta\,dt \\
\theta(t) &= \int_0^t q\,dt - \int_0^t r\phi\,dt \\
\psi(t) &= \int_0^t r\,dt + \int_0^t q\phi\,dt
\end{aligned}
\tag{3-21}
$$

In the above equations, ϕ, θ, and ψ are angles in space and when all are zero, the platform coordinate system coincides with the orientation coordinate system by definition. Having the space angles zero is no guarantee that the integrals of p, q and r are zero. Depending upon the past history of motion, p, q and r may or may not be zero. For example, having $\phi(t) = 0$ does not ensure that $\int_0^t \phi r\,dt$ is zero. A typical platform stabilization system can maintain the integrals of p, q and r at zero. Therefore, the error of the platform coordinate system is given from Eq. (3-21) by

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\Delta \phi(t) = \int_0^t \theta r \, dt$$

$$\Delta \theta(t) = -\int_0^t \phi r \, dt$$

$$\Delta \psi(t) = \int_0^t \phi q \, dt \tag{3-22}$$

If the vibratory motions are small, the integrals above become significant only if there is a rectification of the motions. Let

$$\theta = \theta_o \sin \omega t$$

$$r = r_o \sin(\omega t - \gamma)$$

then

$$\Delta \phi(t) = \theta_o r_o \int_0^t \sin \omega t (\sin \omega t \cos \gamma - \cos \omega t \sin \gamma) \, dt$$

$$= \theta_o r_o \int_0^t (\sin^2 \omega t \cos \gamma - \sin \omega t \cos \omega t \sin \gamma) \, dt$$

$$= \frac{\theta_o r_o}{\omega} \left[\left(\frac{\omega t}{2} - \frac{1}{4} \sin 2\omega t \right) \cos \gamma + \frac{1}{4}(1 + \cos 2\omega t) \sin \gamma \right] \tag{3-23}$$

An examination of Eq. (3-23) reveals one linear term. Therefore, the angle drifts from null with a constant drift rate.

$$\Delta \phi(t) = \frac{\theta_o r_o \cos \gamma}{2}$$

For the system just analyzed, there will exist a coning error that grows with time. One possible method of reducing the error would be by use of a coning computer.[8] The computer would solve Eqs. (3-20) and compensation for the coning drift could then be built into the system.

[illegible]

3.5 Sculling Effect

Sculling produces an acceleration error in accelerometers mounted in gyro stabilized platforms undergoing combined linear and angular motion. This effect will be considered because what is really important is the effect of this error on position indication.

The same coordinate systems and coordinate transformation matrix previously used in describing platform coning apply. Define the total acceleration vector in both coordinate systems as

$$\bar{n} = n_x \bar{i}_p + n_y \bar{j}_p + n_z \bar{k}_p \tag{3-24}$$

$$\bar{n} = n_{x_o} \bar{i}_o + n_{y_o} \bar{j}_o + n_{z_o} \bar{k}_o \tag{3-25}$$

If there is no acceleration error, then

$$\begin{aligned} n_x - n_{x_o} &= 0 \\ n_y - n_{y_o} &= 0 \\ n_z - n_{z_o} &= 0 \end{aligned} \tag{3-26}$$

A check on the above relations will now be made. From transformation (3-16) we obtain

$$\begin{aligned} \bar{n} = n_{x_o} [\bar{i}_p - \psi \bar{j}_p + \theta \bar{k}_p] + n_{y_o} [\psi \bar{i}_p + \bar{j}_p - \phi \bar{k}_p] \\ + n_{z_o} [-\theta \bar{i}_p + \phi \bar{j}_p + \bar{k}_p] \end{aligned} \tag{3-26}$$

Comparison of Eqs. (3-24) and (3-26) produces the relations

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

$$n_x = n_{x_o} + \psi n_{y_o} - \theta n_{z_o}$$

$$n_y = -\psi n_{x_o} + n_{y_o} + \phi n_{z_o}$$

$$n_z = \theta n_{x_o} - \phi n_{y_o} + n_{z_o}$$

and by rearranging

$$n_x - n_{x_o} = \psi n_{y_o} - \theta n_{z_o}$$

$$n_y - n_{y_o} = \phi n_{z_o} - \psi n_{x_o}$$

$$n_z - n_{z_o} = \theta n_{x_o} - \phi n_{y_o} \qquad (3\text{-}27)$$

Equations (3-27), when compared to Eqs. (3-26), show that errors between the accelerations actually measured and the accelerations we desire to measure exist.

Assume a component of the vibration of the platform is at a certain frequency, ω. Then consider any term of Eq. (3-27), say ϕn_{z_o}. Let the linear and angular vibration vary sinusoidally at frequency ω.

If

$$\phi = |\phi| \sin \omega t$$

$$n_{z_o} = |n_{z_o}| \sin(\omega t - \gamma)$$

then

$$\theta n_{z_o} = |\phi||n_{z_o}| \sin \omega t (\sin \omega t - \gamma)$$

$$= \frac{|\phi n_{z_o}|}{2} [(1 - \cos 2\omega t) \cos \gamma - \sin 2\omega t \sin \gamma]$$

which gives a steady state drift of

$$\frac{\left|\phi\, n_{z_o}\right|}{2} \cos \gamma \tag{3-28}$$

As a consequence of the analysis, it is seen that small angular motions of the platform can result in rectified acceleration drifts whose magnitudes depend on the phase relationships between the linear and angular motions. The name "sculling" [9] is given to this effect because the motion causing the drift is identical to that produced in propelling a boat with an oar at the stern.

CHAPTER 4

MAGNITUDE OF THE ANGULAR VIBRATION INACCURACIES

4.1 General

The errors producing forced precession of a single-degree-of-freedom gyroscope have been analyzed in the previous chapter. This chapter deals with a more practical consideration; the magnitudes of the various error terms. When practical oscillations capable of being produced by the Angular Vibrator, built by the Instrumentation Laboratory of the Massachusetts Institute of Technology, will be assumed.

The following are assumed as typical values for a single-degree-of-freedom integrating gyroscope:

$$\Delta I = I_{IA} - I_{SA} = -650 \text{ gm cm}^2$$

$$H \approx H_{SP} = 2 \times 10^6 \frac{\text{gm cm}^2}{\text{sec}}$$

$$\frac{H}{C_d} = 1$$

$$A_{(gim)} \leq .01745 \text{ radians}$$

A comparison of the various inaccuracies previously considered will now be made.

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[illegible]

[illegible]

$$[illegible]$$

[illegible]

4.2 Magnitude Determination

For convenience, the inaccuracies developed in the previous chapter are again listed.

$$M_D = (I_{IA} - I_{SA}) W_{(IC)IA} W_{(IC)SA} + (I_{IA} - I_{SA}) A_{(gim)} (W^2_{(IC)IA} - W^2_{(IC)SA}) \tag{3-8}$$

$$M_{ge} = -H_{SP} (W_{(IC)IA} - A_{(gim)} W_{(IC)SA}) \tag{3-9}$$

$$\begin{aligned} \Delta \phi(t) &= \int_0^t \theta r \, dt \\ \Delta \theta(t) &= - \int_0^t \phi r \, dt \\ \Delta \psi(t) &= \int_0^t \phi q \, dt \end{aligned} \tag{3-22}$$

$$\begin{aligned} n_x - n_{x_o} &= \psi n_{y_o} - \theta n_{z_o} \\ n_y - n_{y_o} &= \phi n_{z_o} - \psi n_{x_o} \\ n_z - n_{z_o} &= \theta n_{x_o} - \phi n_{y_o} \end{aligned} \tag{3-27}$$

Equations (3-8) and (3-9) apply to a single gyroscope instrument while Eqs. (3-22) and (3-27) apply to a platform stabilized system. The magnitudes of the various terms will be determined for the case of harmonic angular velocities.

4.2.1 Magnitude of the Dynamic Torque

Let

$$\begin{aligned} W_{(IC)IA} &= \left| W_{IA} \right| \sin(\omega_1 t + \phi_1) \\ W_{(IC)SA} &= \left| W_{SA} \right| \sin(\omega_2 t + \phi_2) \end{aligned} \tag{4-1}$$

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

The anisoinertia component of Eq. (3-8), when using Eq. (4-1), may be written

$$\frac{1}{2}\,\Delta I\,\left|W_{IA}\right|\,\left|W_{SA}\right|\Big[\cos\left[(\omega_1-\omega_2)t+(\phi_1-\phi_2)\right] - \cos\left[(\omega_1+\omega_2)t+\phi_1+\phi_2\right]\Big] \tag{4-2}$$

If

$$\omega_1=\omega_2=\omega$$

then the above reduces to:

$$\frac{\Delta I}{2}\left|W_{IA}\right|\left|W_{SA}\right|\left[\cos(\phi_1-\phi_2)-\cos(2\omega t+\phi_1+\phi_2)\right] \tag{4-3}$$

Equation (4-3) consists of a constant term, and a cosine term that has a zero average value. Since $(\phi_1-\phi_2)$ represents the phase difference between the two velocities it is evident that only the in phase components of the motions create an error. If the two angular velocities are in quadrature, there will be no resultant error.

If $\omega_1 \neq \omega_2$, then the torque will oscillate at a beat frequency $(\omega_1+\omega_2)$, $(\omega_1-\omega_2)$ and have an average value of zero.

The second component of Eq. (3-8) for $\omega=\omega_1=\omega_2$ may be written as:

$$\frac{\Delta I}{2}A_{(gim)}\Big[\left|W_{IA}\right|^2-\left|W_{SA}\right|^2+\cos 2\omega t\left(\left|W_{SA}\right|^2\cos 2\phi_2 + \left|W_{IA}\right|^2\cos 2\phi_1\right)-\sin 2\omega t\left(\left|W_{SA}\right|^2\sin 2\phi_2 + \left|W_{IA}\right|^2\sin 2\phi_1\right)\Big] \tag{4-4}$$

Before the magnitude can be determined, the variation of gimbal angle with time must be known. For the present, an ideal integrating gyro will be assumed so that

$$\dot{A}_{(gim)} = W_{(IC)IA}$$

and

$$A_{(gim)} = \int W_{(IC)IA}\, dt$$

$$= \frac{-|W_{IA}|}{\omega} \cos(\omega t + \phi_1) \tag{4-5}$$

Substituting (4-5) into (4-4) we get

$$-\frac{\Delta I}{2\omega} |W_{IA}| \sin(\omega t + \phi_1) \Big[|W_{IA}|^2 - |W_{SA}|^2$$
$$+ \cos 2\omega t (|W_{SA}|^2 \cos 2\phi_2 + |W_{IA}|^2 \cos 2\phi_1)$$
$$- \sin 2\omega t (|W_{SA}|^2 \sin 2\phi_2 + |W_{IA}|^2 \sin 2\phi_1) \Big] \tag{4-6}$$

All the terms in Eq. (4-6) have zero as an average value; therefore, the entire expression may be neglected. The same applies for the case of $\omega_1 \neq \omega_2$.

4.2.2 Magnitude of the Gyroscopic Torque

We next turn our attention to Eq. (3-9). After making substitutions for the motions we have

$$M_{ge} = - H_{SP} [|W_{IA}| \sin(\omega_1 t + \phi_1) - A_{(gim)} |W_{SA}| \sin(\omega_2 t + \phi_2)] \tag{4-7}$$

The first term in the bracket will have a zero average value. Our attention will now focus on the cross coupling coning term:

$$A_{(gim)} H_{SP} |W_{SA}| \sin(\omega_2 t + \phi_2) \tag{4-8}$$

Again assuming a perfect gyro so that relation (4-5) applies we obtain

$$\frac{1}{\omega} H_{SP} |W_{IA}| \, |W_{SA}| \cos(\omega_1 t + \phi_1) \sin(\omega_2 t + \phi_2) \tag{4-9}$$

If

$$\omega_1 = \omega_2 = \omega$$

we can write:

$$-\frac{1}{2\omega} H_{SP} |W_{IA}| \, |W_{SA}| \left[\sin(\phi_2 - \phi_1) + \sin(2\omega t + \phi_2 + \phi_1)\right] \tag{4-10}$$

Thus the torque depends on the out of phase components of the velocities. Should the two velocities be in phase there would be no torque. Only velocities in quadrature produce a coning error. From this discussion it is seen that the float drifts from null at a constant rate of

$$\frac{1}{2\omega} |W_{IA}| \, |W_{SA}| \sin(\phi_2 - \phi_1) \quad \text{rad/sec} \tag{4-11}$$

For the condition of $\omega_1 \neq \omega_2$, Eq. (4-9) contains no linear or average torque values.

Note that when coning is a maximum, the anisoinertia effect is a minimum. Angular motions that are not exactly in or out of phase will produce both effects. However, even for this condition it is seen that the anisoinertia error may be neglected, since it is very small in comparison to the cross coupling coning error. This is due to the fact that the coning term involves H_{SP} which is very much larger than any of the values in Eq. (4-3).

The coning drift rate is plotted in Fig. 4-1 as a function of frequency for angular velocities 30, 60 and 90 degrees out of phase.

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$$[illegible]$$

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[illegible]

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[illegible]

[illegible]

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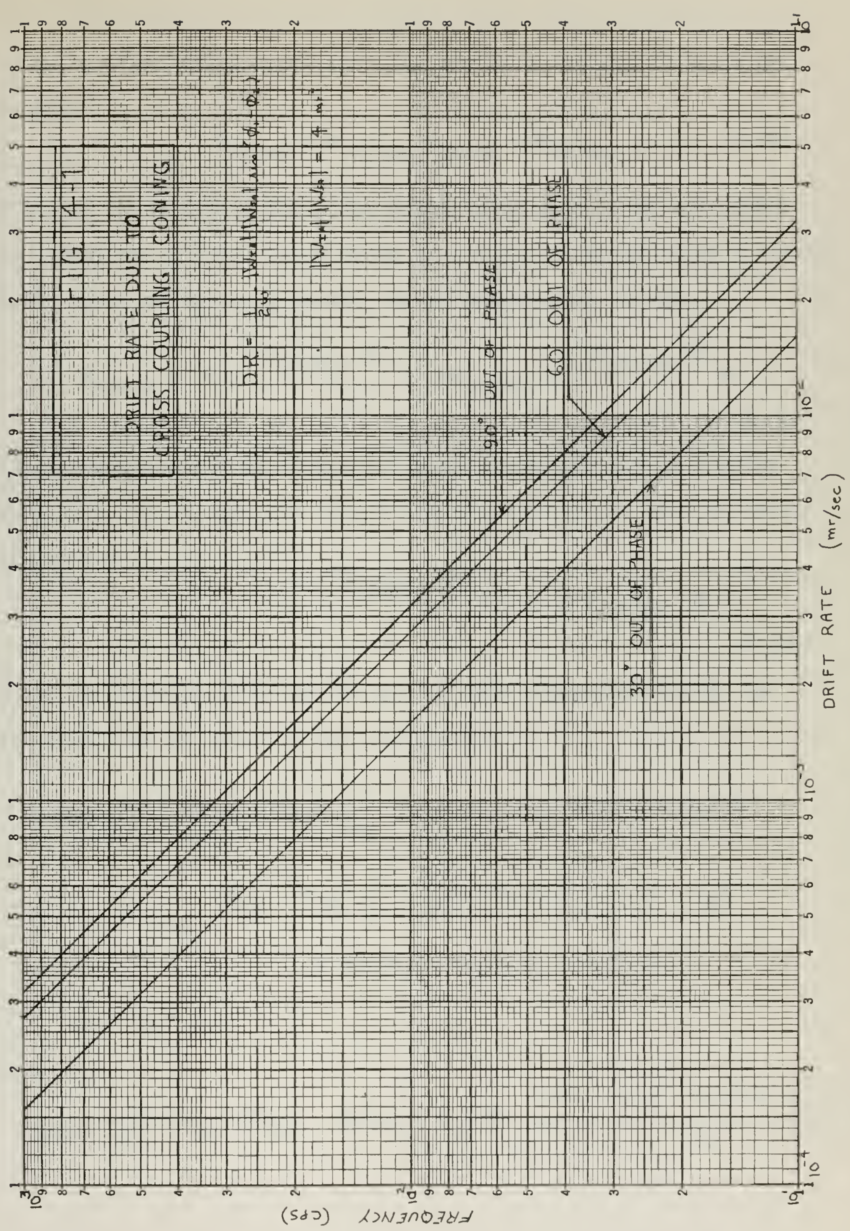
FIG. 4-1
DRIFT RATE DUE TO
CROSS COUPLING CONING
DR = $\frac{1}{2\omega}$ $|W_{x1}||W_{x2}| \sin(\phi_1 - \phi_2)$
$|W_{x1}||W_{x2}| = 4$ mr²
90° OUT OF PHASE
60° OUT OF PHASE
30° OUT OF PHASE
DRIFT RATE (mr/sec)
FREQUENCY (CPS)

All of the previous work has been for an ideal gyroscope. Actually, the true gimbal angle variation should be obtained from the gyro equation

$$\dot{A}_{(gim)} + W_{(IC)SA}\, A_{(gim)} = W_{(IC)IA} \qquad (4\text{-}12)$$

If Eq. (4-12) is used to obtain the gimbal angle motion, the only difference from the previous result would be a nonzero average value of coning drift. This can be easily shown. First integrate Eq. (4-12)

$$A_{(gim)} = e^{-\int W_{(IC)SA}\, dt} \int_0^t W_{(IC)IA}\, e^{\int W_{(IC)SA}\, dt}\, dt \qquad (4\text{-}13)$$

Assuming sinusoidal variations and small magnitudes for the angular velocities, the exponentials may be approximated by the first two terms of a series

$$e^{\pm \int W_{(IC)SA}\, dt} \approx 1 \pm \int W_{(IC)SA}\, dt \qquad (4\text{-}14)$$

Using the relations (4-1) and the gimbal angle expression (4-13), we can determine the nature of the coning term

$$A_{(gim)} \approx \int_0^t |W_{IA}|\, |W_{SA}|\, \frac{1}{\omega_2} \sin(\omega_1 t + \phi_1) \cos(\omega_2 t + \phi_2)\, dt$$

$$\approx \frac{1}{\omega_2} |W_{IA}|\, |W_{SA}|\, \frac{1}{2} \int_0^t \Big[[\sin(\omega_1 t + \phi_1 - \omega_2 t - \phi_2] + \sin[\omega_1 t + \omega_2 t + \phi_1 + \phi_2] \Big]\, dt$$

if $\omega_1 = \omega_2 = \omega$

then

$$A_{(gim)} \approx \frac{1}{2\omega} |W_{IA}|\, |W_{SA}| \quad \sin(\phi_1 - \phi_2)\, xt + \frac{\cos(\phi_1 + \phi_2) - \cos(2\omega t + \phi_1 + \phi_2)}{2\omega}$$

It is seen that a constant drift rate appears similar to expression (4-11) plus an oscillatory term and a nonzero average term. Since the constant drift rate term dominates, the approximation for gimbal angle motion using Eq. (4-5) is valid.

4.2.3 Magnitude of Platform Coning

From Chapter 3 it was seen that platform coning effect causes the Euler angles to drift from null at a rate equivalent to $\phi(t)$

$$\Delta\dot{\phi}(t) = \frac{\theta_o r_o}{2} \cos\gamma \tag{4-15}$$

The above equation was the result of integrating one of the error equations (3-22).

If it is assumed that

$$r = r_o \sin(\omega t + \gamma) \approx \dot{\psi}$$

$$\psi = \psi_o \cos(\omega t + \gamma)$$

then

$$\dot{\psi} = -\psi_o \omega \sin(\omega t + \gamma)$$

$$r_o = -\psi_o \omega$$

and Eq. (4-15) is now

$$\Delta\dot{\phi}(t) = \frac{-\psi_o \theta_o \omega \cos\gamma}{2} \tag{4-16}$$

The assumption will now be made that the amplitudes of the motions are known but that the directions and phase relationships between the components of the motion are random.

Let

$$\psi_o = K \cos\beta$$

$$\theta_o = K \sin\beta$$

Hence
$$\Delta \dot{\phi}(t) = \frac{K^2 \omega}{4} \sin 2\beta \quad \cos \gamma$$

Now assume that β and γ are independent and uniformly distributed from 0 to 2π, so that the mean value of each of the above trigonometric terms is zero, and the variance of each is 1/2. For example

$$\text{Mean } (\cos \gamma) = \mu \cos \gamma = E(\cos \gamma) = \frac{1}{2\pi} \int_0^{2\pi} \cos \gamma \; d\gamma = 0$$

$$\text{Variance } (\cos \gamma) = \sigma^2 (\cos \gamma) = E(\cos^2 \gamma) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \gamma \; d\gamma = 1/2$$

We can now write

$$E[\Delta \dot{\phi}(t)] = \left(\frac{K^2 \omega}{4}\right)^2 E[\sin 2\beta \quad \cos \gamma]^2$$

$$= \sigma^2 = \left(\frac{K^2 \omega}{4}\right)^2 \left[\frac{1}{2} \; \frac{1}{2}\right]$$

and finally the standard deviation or RMS value is

$$\sigma = \frac{K^2 \omega}{8} \tag{4-17}$$

where

K = maximum amplitude, radians

ω = frequency, rad/sec

Equation (4-17) is plotted in Fig. 4-2.

4.2.4 Magnitude of Sculling Effect

The sculling effect will be treated by the method used in the previous section. Consider the first of Eqs. (3-27) :

$$n_x - n_{x_o} = \psi \, n_{y_o} - \theta \, n_{z_o} \tag{4-18}$$

[illegible]

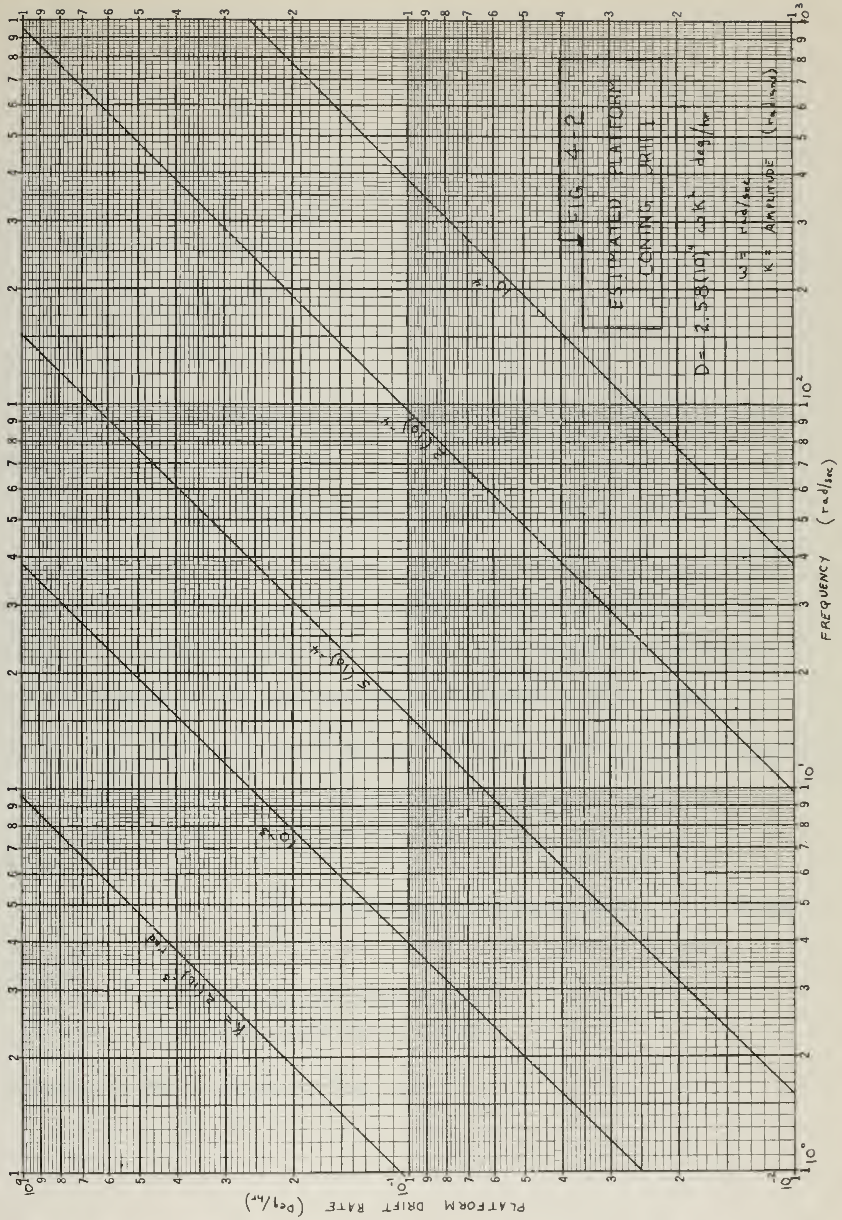
FIG. 4-2
ESTIMATED PLATFORM
CONING DRIFT
D = 2.58(10)^4 ωK^2 deg/hr
ω = rad/sec
K = AMPLITUDE (radians)
FREQUENCY (rad/sec)
PLATFORM DRIFT RATE (Deg/hr)

Using the form of the sculling error found in Chapter 3 we can write

$$n_x - n_{x_0} = \frac{1}{2} \left|\psi_0 \, n_{y_0}\right| \cos \gamma_1 - \frac{1}{2} \left|\theta_0 \, n_{z_0}\right| \cos \gamma_2 \tag{4-19}$$

If we let

$$\left| n_{y_0} \right| = n \cos \beta$$

$$\left| n_{z_0} \right| = n \sin \beta$$

$$\left| \psi_0 \right| = K \cos \alpha$$

$$\left| \theta_0 \right| = K \sin \alpha$$

then

$$n_x - n_{x_0} = \frac{1}{2} nK (\cos \alpha \cos \beta \cos \gamma_1 - \sin \alpha \sin \beta \cos \gamma_2) \tag{4-20}$$

Consider that α, β, γ_1, and γ_2 are independent of each other and uniformly distributed from 0 to 2π. The mean value of each trigonometric function is zero and the variance of each is 1/2. Then

$$E[(n_x - n_{x_0})^2] = (\frac{1}{2} nK)^2 \; E[\cos^2 \alpha \cos^2 \beta \cos^2 \gamma_1 + \sin^2 \alpha \sin^2 \beta \cos^2 \gamma_2]$$

$$\sigma(n_x - n_{x_0}) = \frac{1}{2} nK \left[2 (\frac{1}{2}) (\frac{1}{2}) (\frac{1}{2}) \right]^{1/2}$$

$$= \frac{nK}{4} \quad \text{standard deviation or RMS} \tag{4-20}$$

where

n = maximum acceleration (ft/sec^2)

K = maximum amplitude (radians)

Equation (4-20) is plotted in Fig. 4-3.

[illegible]

[illegible]

[illegible]

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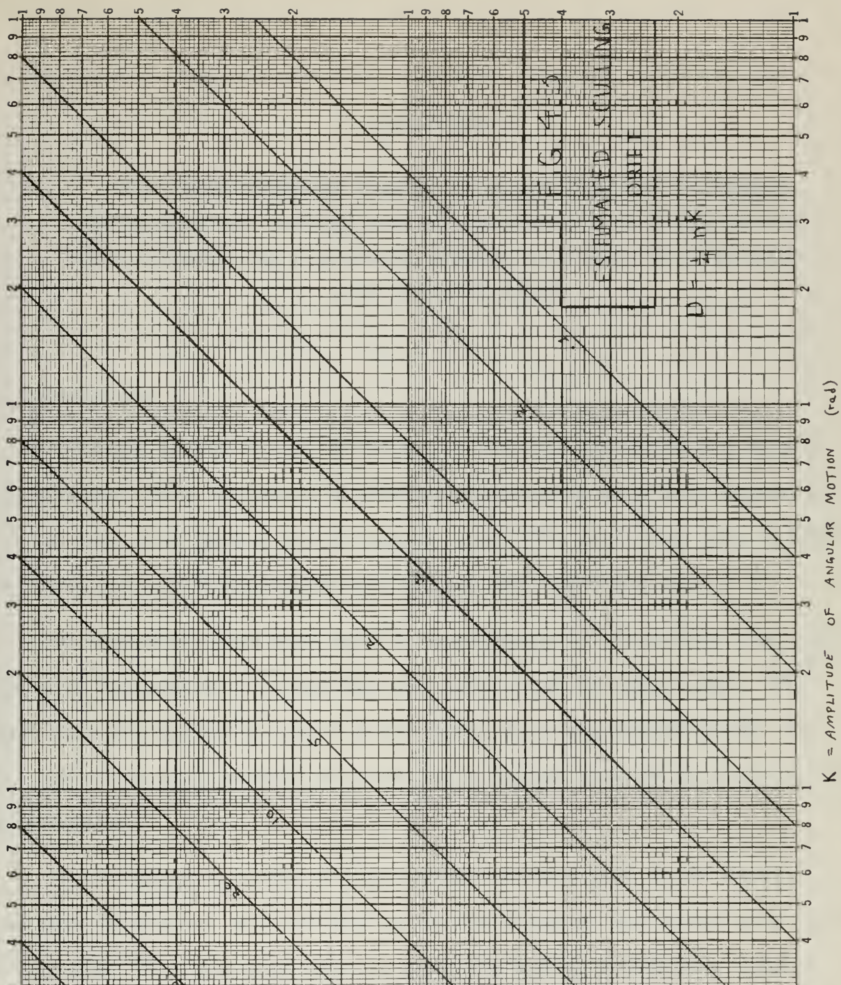
FIG 4.3
ESTIMATED SCULLING DRIFT
D = ½ nK
K = AMPLITUDE OF ANGULAR MOTION (rad)
ACCELERATION DRIFT (g's)
n = 100
50
30
10
5
3
2
1

CHAPTER 5

THE RESPONSE OF THE SINGLE-DEGREE-OF-FREEDOM INTEGRATING GYROSCOPE TO TWO ANGULAR VIBRATIONS SIMULTANEOUSLY APPLIED ABOUT THE SPIN REFERENCE AND OUTPUT AXES

5.1 Introduction

Gyroscopic effects caused by sinusoidal angular vibrations are studied in this chapter. Two vibrations of different frequencies have been applied to two of the body fixed axes of the gyroscope case. The particular axes to which these inputs are applied are the output and the spin reference axis.

A series solution has been used to approximate the theoretical angle which should exist between the float and inertial space about the output axis. The method used was applied to a case of two equal input frequencies in ref. 10. Thus, this is the more general situation and a further study of the problem with varying inputs.

A typical dynamic model of a gyroscope has been used as the basis for the equation of motion. Effects such as inertia damping, and compliance have been included. The resulting equation presently has no closed form solution and the approximate series method was used.

The investigation was an attempt to discover if and when gyroscopic drift can be expected along with the associated values of magnitude. .

5.2 Definition of Symbols

The following symbols and definitions have been used in the analyses: (units are in parentheses)

a = constant (radian)

b = constant (radian)

c = constant (radian)

$c_{d(OA)}$ = damping coefficient between float and case about the output axis (dyne-cm-sec/radian)

$h = \dfrac{H}{c_{d(OA)}}$

$k = \dfrac{K_{(C-F)OA}}{c_{d(OA)}}$

t = time (seconds)

x = the variable representing $A_{(I-F)OA}$

$A_{(I-C)NS}$ = angular rotation of the case about a stationary axis aligned with the earth's axis which coincides with the spin reference axis when $A_{(I-C)OA} = 0$ (radian).

$A_{(I-F)OA}$ = angular rotation of the float about the output axis (radian)

$A_{(I-C)OA}$ = angular rotation of the case about the output axis (radian)

A = constant (radian)

B = constant (radian)

C = constant (radian)

D = constant (radian)

E = constant (radian)

F = constant (radian)

G = constant (radian)

H = angular momentum of the wheel (dyne-cm-sec)

H_1 = constant (radian)

$I_{(F)OA}$ = inertia of the float about the output axis (gm-cm^2)

I_W = inertia of the wheel about a radial axis (gm-cm^2)

J = constant (radian)

K = constant (radian)

$K_{(C-F)OA}$ = elastic restraint about the output axis between float and case. Rotational stiffness between float and case about the output axis (dyne-cm/rad)

L = constant (radian)

M = constant (radian)

N = constant (radian)

ω_1 = circular frequency of vibrational input (rad/sec)

ω_2 = circular frequency of vibrational input (rad/sec)

$$\tau = \frac{I_{(F)OA} + I_W}{c_{d(OA)}} = \text{gyroscope time constant (seconds)}$$

(The inertia of the wheel is added to that of the float in τ since the wheel is assumed to be perfectly rigid with respect to the float about the output axis in the analyses.)

5.3 The Equation of Motion of the Dynamic Model[10]

The simplified equation of motion of the dynamic model shown in Fig. 5-1 is given by :

$$K_{(C-F)OA}\left[A_{(I-F)OA} - A_{(I-C)OA}\right] + c_{d(OA)}\left[\dot{A}_{(I-F)OA} - \dot{A}_{(I-C)OA}\right]$$

$$+ \left[I_{(F)OA} + I_W\right]\ddot{A}_{(I-F)OA} = H\left[\dot{A}_{(I-C)NS}\, A_{(I-F)OA}\right] \quad (5\text{-}1)$$

By using small angle theory, the term $H\left[\dot{A}_{(I-C)NS}\, A_{(I-F)OA}\right]$ is introduced by the component of angular velocity about the input axis produced by the motion about the space-fixed axis. Division by $c_{d(OA)}$ and rearrangement yields:

$$\left[\frac{K_{(C-F)OA}}{c_{d(OA)}}\right] A_{(I-F)OA} + \dot{A}_{(I-F)OA} + \left[\frac{I_{(F)OA} + I_W}{c_{d(OA)}}\right] \ddot{A}_{(I-F)OA}$$

$$+ \left[\frac{-H}{c_{d(OA)}}\right] \dot{A}_{(I-C)NS}\, A_{(I-F)OA} = \left[\frac{K_{(C-F)OA}}{c_{d(OA)}}\right] A_{(I-C)OA} + \dot{A}_{(I-C)OA}$$

or

$$\left[k - h\dot{A}_{(I-C)NS}\right] A_{(I-F)OA} + \dot{A}_{(I-F)OA} + \tau \ddot{A}_{(I-F)OA}$$

$$= \dot{A}_{(I-C)OA} + k A_{(I-C)OA} \qquad (5\text{-}1a)$$

The stiffnesses between the wheel and float about the input and output axes are assumed to be infinite compared to the inertia of the wheel for the frequency range considered. The stiffness and/or damping between the float and case about the input axis is also assumed infinite compared to the inertia of the float and wheel within the frequency range considered.

5.4 Method of Solution of the Equations of Motion for Two Different Sinusoidal Vibration Inputs of Small Magnitude

The inputs introduced to the gyroscope case may be represented as follows:[10]

$$A_{(I-C)NS} = a \sin \omega_1 t \qquad (5\text{-}2)$$

$$A_{(I-C)OA} = b \sin \omega_2 t + c \cos \omega_2 t \qquad (5\text{-}3)$$

In most applications of the single-degree-of-freedom integrating gyroscope, the constant h is designed equal to unity. For purposes of simplification, this constant will be included in the vibration inputs as a factor of the constant a.

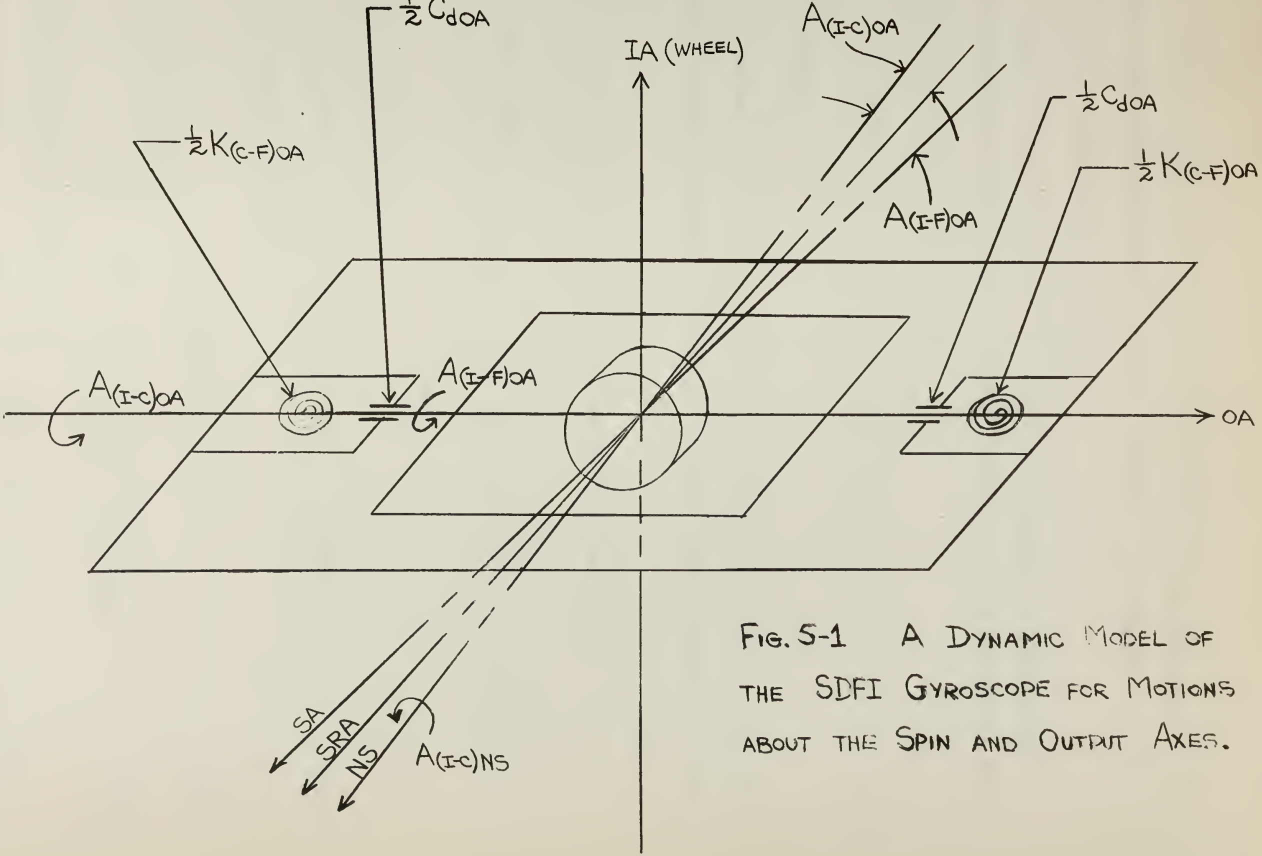

FIG. 5-1 A DYNAMIC MODEL OF THE SDFI GYROSCOPE FOR MOTIONS ABOUT THE SPIN AND OUTPUT AXES.

To be strictly correct, earth rate should be included in the definition of the vibration input about the North-South axis. This rate is, however, small compared to the input angular velocity.

Substitution of equations (5-2) and (5-3) into equation (5-1a) yields a linear differential equation with a variable coefficient of the following form:

$$\left[k - a\omega_1 \cos \omega_1 t\right] x + \dot{x} + \tau \ddot{x}$$
$$= b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \tag{5-4}$$

where $A_{(I-F)OA} = x$

There is no known closed form solution for equation (5-4). However, if the constants are small compared to unity, a suitable series solution may be obtained.*

5.4.1 Negligible Elastic Restraint Between the Float and Case About the Output Axis

When the elastic restraint is very small compared to the damping constant, i. e.,

$$k \rightarrow 0$$

equation (5-4) reduces to:

$$(-a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \tag{5-5}$$

Assume a solution of the form:

$$x = x_1 + x_2 \tag{5-6}$$

where x_1 is defined by:

$$\dot{x}_1 + \tau \ddot{x}_1 = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \tag{5-7}$$

Substitute (5-6) and (5-7) into (5-5):

$$(-a\omega_1 \cos \omega_1 t)(x_1 + x_2) + (\dot{x}_1 + \dot{x}_2) + \tau(\ddot{x}_1 + \ddot{x}_2) = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \tag{5-8}$$

*The series solution technique was suggested by Mr. Daniel Goldenberg of the Mathematics Group, MIT Instrumentation Laboratory.

or

$$(-a\omega_1 \cos \omega_1 t)x_2 + \dot{x}_2 + \tau \ddot{x}_2 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad (5\text{-}8a)$$

Now let $x_2 = x_3 + x_4$ (5-9)

where x_3 is defined by:

$$\dot{x}_3 + \tau \ddot{x}_3 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad (5\text{-}10)$$

Substitute (5-9) and (5-10) into (5-8a):

$$(-a\omega_1 \cos \omega_1 t)(x_3 + x_4) + (\dot{x}_3 + \dot{x}_4) + \tau(\ddot{x}_3 + \ddot{x}_4) = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad (5\text{-}11)$$

or

$$(-a\omega_1 \cos \omega_1 t)\, x_4 + \dot{x}_4 + \tau \ddot{x}_4 = (a\omega_1 \cos \omega_1 t)\, x_3 \qquad (5\text{-}11a)$$

Continuation of this process results in a series solution of the following form:

$$x = x_1 + x_3 + x_5 + \ldots\ldots\ldots\ x_{2m-1} + x_{2m} \qquad (5\text{-}12)$$

where x_{2m} is the even term remaining after the above process is repeated m times. x_{2m} satisfies the equation:

$$(-a\omega_1 \cos \omega_1 t)\, x_{2m} + \dot{x}_{2m} + \tau \ddot{x}_{2m} = (a\omega_1 \cos \omega_1 t)\, x_{2m-1} \qquad (5\text{-}13)$$

A Brief Summary for $k \rightarrow 0$ is:

Problem:

$$(-a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \qquad (5\text{-}5)$$

let

$$x = x_1 + x_3 + x_5 + \ldots\ x_{2m-1} + x_{2m} \qquad (5\text{-}12)$$

where:

$$\dot{x}_1 + \tau \ddot{x}_1 = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \qquad (5\text{-}7)$$

$$\dot{x}_3 + \tau \ddot{x}_3 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad (5\text{-}10)$$

$$\dot{x}_5 + \tau \ddot{x}_5 = (a\omega_1 \cos \omega_1 t)\, x_3 \qquad (A\text{-}6)$$

$$\dot{x}_{(2N+1)} + \tau \ddot{x}_{(2N+1)} = (a\omega_1 \cos \omega_1 t)\, x_{(2N+1)} \qquad (A\text{-}6a)$$

Each successive term in the series of equation (5-12) can be shown to be smaller than the preceding term by a factor of approximately a. (Refer to ref. 10 for further discussion.) Since a is of the order of 2×10^{-3} radians, this series is seen to converge rapidly. Hence all terms beyond x_3 can be reasonably neglected.

The following evaluation of the series and associated coefficients clearly shows the appearance of the factor a which permits terms beyond a certain point to be neglected. (See Appendix A for solution details.)

$A_{(I-F)OA} = x \qquad k = \text{elastic restraint} = 0$

$$x = x_1 + x_3 + x_5 + \text{other terms which are neglected} \tag{5-12}$$

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \tag{A-7}$$

$$x_3 = C \sin \omega_3 t + D \sin \omega_4 t + E \cos \omega_3 t + F \cos \omega_4 t \tag{A-17}$$

$$\omega_3 = \omega_1 + \omega_2 \qquad \omega_4 = \omega_1 - \omega_2$$

$$\begin{aligned} x_5 = {} & G \sin(2\omega_1 t + \omega_2 t) + H_1 \cos(2\omega_1 t + \omega_2 t) + J \sin(2\omega_1 t - \omega_2) \\ & + K \cos(2\omega_1 t - \omega_2 t) + L \sin(-\omega_2 t) + M \cos(-\omega_2 t) \\ & + N \sin(\omega_2 t) + P \cos(\omega_2 t) \end{aligned} \tag{A-37}$$

$$A = \frac{b + c\tau\omega_2}{1 + \tau^2 \omega_2^2} \tag{A-12}$$

$$B = \frac{c - b\tau\omega_2}{1 + \tau^2 \omega_2^2} \tag{A-14}$$

$$C = \frac{a\omega_1 (B - \tau A \omega_3)}{2\omega_3 (1 + \tau^2 \omega_3^2)} \tag{A-24}$$

$$D = \frac{a\omega_1 (B + \tau A \omega_4)}{2\omega_4 (1 + \tau^2 \omega_4^2)} \tag{A-28}$$

$$E = \frac{-a\omega_1 (A + \tau B \omega_3)}{2\omega_3 (1 + \tau^2 \omega_3^2)} \tag{A-26}$$

$$F = \frac{a\omega_1 (A - \tau B \omega_4)}{2\omega_4 (1 + \tau^2 \omega_4^2)} \tag{A-30}$$

$$G = \frac{a\omega_1[E - \tau C(2\omega_1 + \omega_2)]}{2(2\omega_1 + \omega_2)[1 + \tau^2(2\omega_1 + \omega_2)^2]}$$

(A-47)

$$H_1 = \frac{-a\omega_1[C + \tau E(2\omega_1 + \omega_2)]}{2(2\omega_1 + \omega_2)[1 + \tau^2(2\omega_1 + \omega_2)^2]}$$

(A-48)

$$J = \frac{a\omega_1[F - \tau D(2\omega_1 - \omega_2)]}{2(2\omega_1 - \omega_2)[1 + \tau^2(2\omega_1 - \omega_2)^2]}$$

(A-50)

$$K = \frac{-a\omega_1[D + \tau F(2\omega_1 - \omega_2)]}{2(2\omega_1 - \omega_2)[1 + \tau^2(2\omega_1 - \omega_2)^2]}$$

(A-49)

$$L = \frac{a\omega_1[-E + \tau C\omega_2]}{2\omega_2[1 + \tau^2\omega_2^2]} \qquad \text{(A-51)}$$

$$M = \frac{-a\omega_1[C + \tau E\omega_2]}{2\omega_2[1 + \tau^2\omega_2^2]} \qquad \text{(A-52)}$$

$$N = \frac{a\omega_1[F + \tau D\omega_2]}{2\omega_2[1 + \tau^2\omega_2^2]} \qquad \text{(A-53)}$$

$$P = \frac{a\omega_1[D - \tau F\omega_2]}{2\omega_2[1 + \tau^2\omega_2^2]} \qquad \text{(A-54)}$$

$$\left.\begin{aligned} \tau &= .0009\ \text{sec} = .0 \times 10^{-3}\text{sec} \\ a_{max} &= 2 \times 10^{-3}\ \text{radians} \\ (\sqrt{b^2+c^2})_{max} &= 4 \times 10^{-3}\ \text{radians} \end{aligned}\right\}$$

Typical values as used by the MIT Instrumentation Laboratory Angular Shaker

With negligible elastic restraint between the float and case, the series solution consists of sinusoids. Such sinusoids will exhibit a time-average value of zero except for special cases which are of interest.

The first case has been studied in ref. 10 and concerns equal frequencies,

$$\omega_1 = \omega_2$$

As in further explained in Appendix A, equal input frequencies will cause the terms of D sin $\omega_4 t$ and F cos $\omega_4 t$ to have indeterminate values. By applying

L'Hospital's Rule, a constant drift term which is time dependent is found to exist. This term is:

$$\text{Drift} = \left[\frac{Ba\omega}{2}\right] t \qquad \text{where } \omega = \omega_1 = \omega_2$$

This existing drift is identical to that found in ref. 10.

The second case of interest concerns the almost natural region for suspicion, namely harmonics. Terms concerning x_5 of the series give indeterminate values when $2\omega_1 = \omega_2$. By L' Hospital's Rule, a drift term appears which is very similar to the term arising from equal frequency input. This term is:

$$\text{Drift} = \left[\frac{Fa\omega_1}{2}\right] t$$

See Appendix A for proof and necessary mathematics

$$2\omega_1 = \omega_2$$

If further series terms such as x_7 and x_9 were evaluated, drift terms associated with other harmonic multiples are found. Thus, the conclusion is that equal frequencies and harmonics are capable of causing drift for the case of negligible elastic restraint.

The physical case will probably display many superimposed input frequencies. This is likely to have some harmonics thereby creating drift. Such drift is substantiated by this theoretical solution but it is suspected that frequencies approaching the harmonics may well create actual physical drift.

5.4.2 Finite Elastic Restraint Between the Float and Case About the Output Axis

For the case when $k \neq 0$, it is desired to find the solution of the following equation:

$$(k - a\omega_1 \cos\omega_1 t)\,x + \dot{x} + \tau\ddot{x} = b\omega_2 \cos\omega_2 t - c\omega_2 \sin\omega_2 t + kb \sin\omega_2 t + kc\cos\omega_2 t \tag{5-4}$$

The mathematical solution with details is in Appendix A. The resulting series with values of coefficients in terms of the known constants is as follows:

$$A_{(I-F)OA} = x \qquad k \neq 0$$

$$x = x_1 + x_3 + x_5 + \text{ other terms which are neglected} \tag{5-12}$$

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \tag{A-67}$$

$$x_3 = D \sin \omega_1 t \sin \omega_2 t + E \sin \omega_1 t \cos \omega_2 t + F \cos \omega_1 t \sin \omega_2 t + G \cos \omega_1 t \cos \omega_2 t \tag{A-75}$$

$$x_5 = H_1 \sin 2\omega_1 t \sin \omega_2 t + I \sin 2\omega_1 t \cos \omega_2 t + J \cos 2\omega_1 t \sin \omega_2 t + L \cos 2\omega_1 t \cos \omega_2 t + M \sin \omega_2 t + N \cos \omega_2 t \tag{A-89}$$

Expressions for the coefficients in terms of known constants may be found in Appendix A. The coefficients have been left in determinant form.

By the use of suitable trigonometry relations the following forms are obtained for x_3:

$$\begin{aligned} x_3 = \; & \frac{D}{2}[\cos(\omega_1 - \omega_2)t] - \frac{D}{2}[\cos(\omega_1 + \omega_2)t] \\ & + \frac{E}{2}[\sin(\omega_1 + \omega_2)t] + \frac{E}{2}[\sin(\omega_1 - \omega_2)t] \\ & + \frac{F}{2}[\sin(\omega_1 + \omega_2)t] - \frac{F}{2}[\sin(\omega_1 - \omega_2)t] \\ & + \frac{G}{2}[\cos(\omega_1 + \omega_2)t] + \frac{G}{2}[\cos(\omega_1 - \omega_2)t] \end{aligned} \tag{A-75a}$$

When $\omega_1 = \omega_2$, the case of ref. 10 occurs. An indeterminate form does not appear because k is finite. Thus time dependent drifting is not a factor with non-zero elastic restraint. However, this special case of equal frequencies gives a value of unity to terms containing $\cos(\omega_1 - \omega_2)t$ and the result is a steady non-zero value for the angle between the float and inertial space about the output axis. This is verified by ref. 10.

[illegible]

The other case of interest with finite elastic restraint again concerns harmonics. This can easily be seen by converting x_5 terms by means of trigonometry relations to the following:

$$
\begin{aligned}
x_5 = \quad & \frac{H_1}{2}\,[\cos(2\omega_1 - \omega_2)t] \; - \; \frac{H_1}{2}\,[\cos(2\omega_1 + \omega_2)t] \\
& + \frac{I}{2}\,[\sin(2\omega_1 + \omega_2)t] \; + \; \frac{I}{2}\,[\sin(2\omega_1 - \omega_1)t] \\
& + \frac{J}{2}\,[\sin(2\omega_1 + \omega_2)t] \; - \; \frac{J}{2}\,[\sin(2\omega_1 - \omega_2)t] \\
& + \frac{L}{2}\,[\cos(2\omega_1 + \omega_2)t] \; + \; \frac{L}{2}\,[\cos(2\omega_1 - \omega_2)t] \\
& + M \sin \omega_2 t \; + \; N \cos \omega_2 t
\end{aligned}
\qquad \text{(A-89a)}
$$

The drift resulting from the harmonic when $2\omega_1 = \omega_2$ is clearly seen as a possibility. However, because of a finite value for elastic restraint, indeterminate forms do not appear. Nevertheless, this particular condition causes an angle to take place about the output axis between the float and inertial space. This occurs when $\cos(2\omega_1 - \omega_2)t$ is unity in value.

This case gives a term showing that a constant angular displacement occurs for the harmonic condition instead of a time dependent drift term.

Such a condition is almost a natural situation for a gyroscope. As the finite restraint is increased, the displacement between the float and the reference in inertial space should become smaller. The extreme limits are for zero elastic restraint and infinite elastic restraint. The former case gives a time dependent drift while $k = \infty$ creates a situation permitting no angle, whatsoever, to exist between the float and inertial reference.

The most interest undoubtedly lies with the case of finite, non-zero, elastic restraint. This is the existing case of practical importance. The

[illegible]

concluding results are that 1) a time-independent drift term is possible if input frequencies contain harmonics or equal values, 2) float angular displacement with reference to inertial space depends on the degree of elastic restraint in the gyroscope, and 3) high harmonic terms may possibly be neglected because of their small value which becomes insignificant when compared to manufacturing tolerances.

It is pointed out that a drift may appear to be present at any one instant of time if a time-averaged measurement is not obtained even if harmonics are not present. This could be of noticeable magnitude dependent upon the input frequencies and their own absolute magnitudes.

It is strongly recommended that this theory be applied to the MIT Instrumentation Laboratory angular shaker in an attempt to verify the results. Harmonic inputs appear feasibile for the shaker by minor adaptation after sufficient data has been collected concerning the case of two inputs of equal frequency.

Terms in the approximate series solution beyond x_5 have only theoretical interest at present. Until gyroscopic tolerances exceed their present values, common sense tells us that the selected series is as good for analysis as would be the unknown closed form solution for the equation of motion.

5.5 Drift of a Typical Gyroscope for the Case of Negligible Elastic Restraint About the Output Axis with Two Equal Input Frequencies

The ensuing results are based on the series solution for the equation of motion when the elastic restraint between the case and float about the output axis is negligible. A further stipulation is that $\omega_1 = \omega_2$ in order that this information may apply directly to the MIT Instrumentation Laboratory angular shaker.

The angular shaker has a frequency range from twenty (20) cycles per second to one hundred (100) cycles per second. The constant "a" has a maximum value of .002 radians and the constants "b" and "c" are limited such that $\sqrt{b^2 + c^2}$ has a maximum of .004 radians. A typical value of τ, the time constant of the damper-mass system, is .0009 seconds.

By definition:

$$A_{(I-C)NS} = a \sin \omega_1 t \qquad (5\text{-}2)$$

$$A_{(I-C)OA} = b \sin \omega_2 t + c \cos \omega_2 t \qquad (5\text{-}3)$$

For this particular case, $\omega = \omega_1 = \omega_2$ as the angular vibrator is capable of applying only one frequency at a time to the two input axes.

If $c = 0$, $A_{(I-C)NS}$ and $A_{(I-C)OA}$ are in phase and they will differ only by the magnitudes of "a" and "b". ($a = b$ is a special case.) In general, "b" and "c" will have finite values thus causing $A_{(I-C)NS}$ and $A_{(I-C)OA}$ to differ by a phase angle as well as magnitude. (Here, again, $a = \sqrt{b^2 + c^2}$ is a special case.)

$$b \sin \omega t + c \cos \omega t = \left[\sin(\omega t + \cot^{-1} b/c)\right] \sqrt{b^2 + c^2} \qquad (5\text{-}14)$$

$$\cot^{-1} b/c = \phi, \text{ a phase angle}$$

If $\omega_1 = \omega_2$, the following is drift rate for $k \quad 0$:

$$\text{Drift Rate} = \frac{ac\omega}{2}\left[\frac{1 - \frac{b}{c}\tau\omega}{1 + \tau^2\omega^2}\right] \qquad (5\text{-}15)$$

A plot is made for drift rate / ac for various ratios of b/c.

$$\frac{\text{Drift Rate}}{ac} = \pi f \left[\frac{1 - \frac{b}{c}\tau\omega}{1 + \tau^2\omega^2}\right] = \frac{\omega}{2}\left[\frac{1 - \frac{b}{c}\tau\omega}{1 + \tau^2\omega^2}\right] \qquad (5\text{-}16)$$

Positive and negative values of b/c were selected to correspond to phase angles from approximately $\pm 5°$ to $\pm 84°$. Ratios for b/c selected are indicated on the graph.

The curves of Fig. 5-2 show that for a frequency range of 20 to 100 cycles per second, the drift rate will undergo a change in direction for $b/c = +2.0$ around 90 cycles per second. The frequency required for this reversal is further decreased for increasing values of b/c.

Although the curves represent specific values for constants, it is believed to be typical of gyroscopes which may be analyzed by the MIT angular shaker. It is suggested that these curves be compared to actual data from the vibrator test machine at the earliest opportunity.

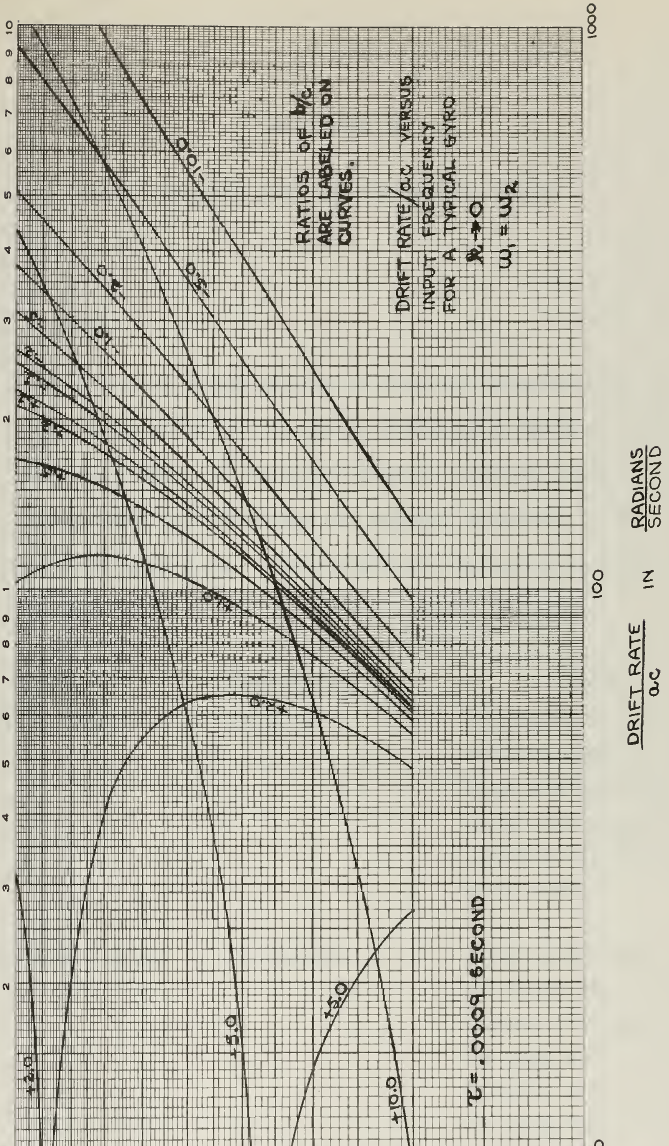
RATIOS OF b/a.c ARE LABELED ON CURVES.
DRIFT RATE/a.c VERSUS INPUT FREQUENCY FOR A TYPICAL GYRO
R → O
ω1 = ω2
τ = .0009 SECOND
FREQUENCY CPS x 10^-1
DRIFT RATE / a c IN RADIANS / SECOND

FIG. 5-2

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

1. When a floated single-degree-of-freedom integrating gyroscope is subjected to vibrations undesired torques or drift rates may exist. A summary of such unwanted errors, arising when the vibrations are angular in nature, is presented in table form. The errors are classified by name, type and cause.

Name	Type Error	Cause
Anisoinertia effect	Dynamic	Simultaneous angular motions about the <u>case</u> input and spin axes along with unequal moments of inertia about these axes.
Cross coupling coning	Gyroscopic	<u>Float</u> precession about its output axis which causes a component of the <u>case</u> spin axis angular velocity to be coupled to the <u>float</u> input axis angular velocity.

Platform coning or gimbal walk	Geometric	Simultaneous angular velocities about platform axes such that the float input axis undergoes a periodic conical motion
Sculling effect	Geometric	Simultaneous angular motion about any platform axis and linear acceleration along a perpendicular axis producing a rectified acceleration along an axis perpendicular to both axes.

2. A second table is presented to illustrate which effects are present when a single-degree-of-freedom integrating gyroscope is subjected to constant angular velocities and periodic vibrations which may be in or out of phase.

	Constant Angular Velocities	Periodic In Phase Angular Velocities	Periodic Out of Phase Angular Velocities
Anisoinertia	Yes	Yes	No
Cross coupling coning	Yes	No	Yes
Platform coning or gimbal walk	Yes	No	Yes
Sculling*	Yes	Yes	No

*For sculling the phase refers to the phase relation between the linear acceleration and the Euler angle motion (space angle motion).

3. Coning is a cause of drift for a single-degree-of-freedom integrating gyroscope.

4. Cross coupling coning is inversely proportional to frequency and directly proportional to the product of the amplitudes of the angular motions.

5. Cross coupling coning may be minimized by use of a tight servo loop which keeps the gimbal angle close to zero.

6. Platform coning is directly proportional to frequency and directly proportional to the square of the amplitude of angular motion.

7. Sculling generated drift is directly proportional to the amplitude of the angular motion and linear acceleration.

8. For zero elastic restraint between the float and case about the output axis: (Based on a series solution of a dynamic model)
   (a) Constant drift rate appears when two equal frequencies are simultaneously applied to the spin and output axes (e.g., $\omega_1 = \omega_2$)
   (b) Constant drift rate appears when the frequencies applied simultaneously to the spin and output axes have a harmonic relation (e.g., $2\omega_1 = \omega_2$)
   (c) The total drift at any instant of time consists of several superimposed sinusoids. Except for the special cases concerning inputs of equal or harmonic frequencies referred to in (a) and (b), the time averaged value of angular displacement of the float with respect to inertial space about the output axis is zero.

9. For finite elastic restraint between the float and case about the output axis: (Based on a series solution of a dynamic model)

   (a) Drift terms which are time independent appear when two equal frequencies are simultaneously applied to the spin and output axes (e. g., $\omega_1 = \omega_2$). These drift terms have a constant magnitude dependent upon a, b, c, τ, ω_1, and ω_2. but not dependent upon time.

   (b) Drift terms which are time independent appear when the frequencies applied simultaneously to the spin and output axes have a harmonic relation (e. g., $2\omega_1 = \omega_2$). These drift terms have a constant magnitude dependent upon a, b, c, τ, ω_1, and ω_2 but not dependent upon time.

   (c) The total drift at any instant of time consists of several superimposed sinusoids. Except for the special cases concerning inputs of equal or harmonic frequencies referred to in (a) and (b), the time averaged value of angular displacement of the float with respect to inertial space about the output axis is zero.

6. 2 Recommendations

1. Verify theoretical results by use of the MIT precision angular vibrator. This should be done for various values of elastic restraint.

2. The present vibrator is capable of applying two simultaneous inputs of equal frequency to two orthogonal gyroscope axes. It is suggested that the machine be investigated to determine if input frequencies with harmonic relationships can be simultaneously applied.

3. Linearity should permit results obtained from a specific frequency applied to one axis to be superimposed upon the results obtained from a different frequency applied to another gyroscope axis. With this procedure, it may be possible to obtain physical data concerning simultaneous inputs of any desired frequencies within the limit of the vibrator. Information for assorted combinations of input frequencies to two orthogonal gyroscope axes could thus be obtained from an angular vibrator capable of applying only one frequency at a time.

4. Great care should be taken to obtain valid data in the low frequency range. This is the suspected source of many uncertainties and may be caused by the relatively obscure bending modes of large missiles and high speed aircraft.

[illegible]

[illegible]

[illegible]

APPENDIX A

SERIES SOLUTION FOR THE EQUATION OF MOTION OF A SINGLE-DEGREE-OF-FREEDOM INTEGRATING GYROSCOPE

A. 1 Series Solution for Negligible Elastic Restraint Between the Float and Case About the Output Axis $(k \to 0)$

The general equation is:

$$(k - a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \qquad \text{(A-1)}$$

If $k \to 0$, we have:

$$(-a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \qquad \text{(A-2)}$$

In accordance with the series established in Section 5. 5, the interest lies with the following terms:

$$x = x_1 + x_3 + x_5 + \dots + x_{2m-1} + x_{2m} \qquad \text{(A-3)}$$

where:

$$\dot{x}_1 + \tau \ddot{x}_1 = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \qquad \text{(A-4)}$$

$$\dot{x}_3 + \tau \ddot{x}_3 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad \text{(A-5)}$$

$$\dot{x}_5 + \tau \ddot{x}_5 = (a\omega_1 \cos \omega_1 t)\, x_3 \qquad \text{(A-6)}$$

$$\dot{x}_{(2N+1)} + \tau \ddot{x}_{(2N+1)} = (a\omega_1 \cos \omega_1 t)\, x_{(2N+1)} \qquad \text{(A-6a)}$$

Let

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \qquad \text{(A-7)}$$

$$\dot{x}_1 = A\, \omega_2 \cos \omega_2 t - B\, \omega_2 \sin \omega_2 t \qquad \text{(A-7a)}$$

$$\ddot{x}_1 = -A\, \omega_2^2 \sin \omega_2 t - B\, \omega_2^2 \cos \omega_2 t \qquad \text{(A-7b)}$$

Substitute (A-7), (A-7a), and (A-7b) into (A-4)

$$[A\omega_2 \cos \omega_2 t - B\omega_2 \sin \omega_2 t] + \tau[-A\omega_2^2 \sin \omega_2 t - B\omega_2^2 \cos \omega_2 t]$$

$$= b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t \qquad \text{(A-8)}$$

Equate coefficients of like terms and solve for A and B.

$$A\omega_2 - \tau B\omega_2^2 = b\omega_2 \qquad \text{(from cosine terms)} \qquad \text{(A-9)}$$

$$-B\omega_2 - \tau A\omega_2^2 = -c\omega_2 \qquad \text{(from sine terms)} \qquad \text{(A-10)}$$

$$A = b + \tau B\omega_2 \qquad \text{(A-9a)}$$

$$B = c - \tau A\omega_2 \qquad \text{(A-10a)}$$

$$A = b + \tau\omega_2[c - \tau A\omega_2] \qquad \text{(A-11)}$$

$$A + \tau^2\omega_2^2 A = b + c\tau\omega_2$$

$$\boxed{A = \frac{b + c\tau\omega_2}{1 + \tau^2\omega_2^2}} \qquad \text{(A-12)}$$

$$B = c - \tau\omega_2[b + \tau B\omega_2] \qquad \text{(A-13)}$$

$$B + \tau^2\omega_2^2 B = c - b\tau\omega_2$$

$$\boxed{B = \frac{c - b\tau\omega_2}{1 + \tau^2\omega_2^2}} \qquad \text{(A-14)}$$

Only those terms affecting the steady state solution are considered.

An equation for x_3 is obtained by substituting (A-7) into (A-5).

$$\dot{x}_3 + \tau\ddot{x}_3 = (a\omega_1 \cos \omega_1 t)(A \sin \omega_2 t + B \cos \omega_2 t) \qquad \text{(A-15)}$$

$$\dot{x}_3 + \tau\ddot{x}_3 = Aa\omega_1 \cos \omega_1 t \sin \omega_2 t + Ba\omega_1 \cos \omega_1 t \cos \omega_2 t \qquad \text{(A-15a)}$$

From basic trigonometry relations, the following are obtained:

$$\cos \omega_1 t \sin \omega_2 t = \frac{1}{2} [\sin(\omega_1 t + \omega_2 t) - \sin(\omega_1 t - \omega_2 t)]$$

$$\cos \omega_1 t \cos \omega_2 t = \frac{1}{2} [\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t)]$$

Let $\omega_3 t = (\omega_1 t + \omega_2 t)$ and $\omega_4 t = (\omega_1 t - \omega_2 t)$

In view of these relations:

$$\dot{x}_3 + \tau \ddot{x}_3 = \frac{Aa\omega_1}{2} [\sin \omega_3 t - \sin \omega_4 t] + \frac{Ba\omega_1}{2} [\cos \omega_3 t + \cos \omega_4 t] \quad \text{(A-16)}$$

Let

$$x_3 = C \sin \omega_3 t + D \sin \omega_4 t + E \cos \omega_3 t + F \cos \omega_4 t \quad \text{(A-17)}$$

$$\dot{x}_3 = C\omega_3 \cos \omega_3 t + D\omega_4 \cos \omega_4 t - E\omega_3 \sin \omega_3 t - F\omega_4 \sin \omega_4 t \quad \text{(A-17a)}$$

$$\ddot{x}_3 = -C\omega_3^2 \sin \omega_3 t - D\omega_4^2 \sin \omega_4 t - E\omega_3^2 \cos \omega_3 t - F\omega_4^2 \cos \omega_4 t \quad \text{(A-17b)}$$

Substitute (A-17), (A-17a), and (A-17b) into (A-16) .

$$[C\omega_3 \cos \omega_3 t + D\omega_4 \cos \omega_4 t - E\omega_3 \sin \omega_3 t - F\omega_4 \sin \omega_4 t]$$

$$+ [-\tau C\omega_3^2 \sin \omega_3 t - \tau D\omega_4^2 \sin \omega_4 t - \tau E\omega_3^2 \cos \omega_3 t - \tau F\omega_4^2 \cos \omega_4 t]$$

$$= \frac{Aa\omega_1}{2} \sin \omega_3 t - \frac{Aa\omega_1}{2} \sin \omega_4 t + \frac{Ba\omega_1}{2} \cos \omega_3 t + \frac{Ba\omega_1}{2} \cos \omega_4 t \quad \text{(A-18)}$$

Equate coefficients of like terms and solve for C, D, E, and F.

$$\cos \omega_3 t: \quad C\omega_3 - \tau E\omega_3^2 = \frac{Ba\omega_1}{2} \quad \text{(A-19)}$$

$$\cos \omega_4 t: \quad D\omega_4 - \tau F\omega_4^2 = \frac{Ba\omega_1}{2} \quad \text{(A-20)}$$

$$\sin \omega_3 t: \quad -E\omega_3 - \tau C\omega_3^2 = \frac{Aa\omega_1}{2} \quad \text{(A-21)}$$

$$\sin \omega_4 t: \qquad -F\omega_4 - \tau D\omega_4^2 = -\frac{Aa\omega_1}{2} \tag{A-22}$$

$$C = \frac{1/2\ Ba\omega_1 + \tau E\omega_3^2}{\omega_3} \tag{A-19a}$$

$$E = \frac{-1/2\ Aa\omega_1 - \tau C\omega_3^2}{\omega_3} \tag{A-21a}$$

Substitute (A-21a) into (A-19)

$$C\omega_3 - \tau\omega_3^2 \left[\frac{-1/2\ Aa\omega_1 - \tau C\omega_3^2}{\omega_3}\right] = \frac{Ba\omega_1}{2} \tag{A-23}$$

$$C\omega_3 + 1/2\,\tau Aa\omega_1\omega_3 + \tau^2 C\omega_3^3 = 1/2\ Ba\omega_1$$

$$C(\omega_3 + \tau^2\omega_3^3) = 1/2\ Ba\omega_1 - 1/2\ \tau Aa\omega_1\omega_3$$

$$\boxed{C = \frac{Ba\omega_1 - \tau Aa\omega_1\omega_3}{2(\omega_3 + \tau^2\omega_3^3)}} \tag{A-24}$$

Substitute (A-19a) into (A-21)

$$-E\omega_3 - \tau\omega_3^2 \left[\frac{1/2\ Ba\omega_1 + \tau E\omega_3^2}{\omega_3}\right] = \frac{Aa\omega_1}{2} \tag{A-25}$$

$$-E\omega_3 - 1/2\,\tau Ba\omega_1\omega_3 - \tau^2 E\omega_3^3 = 1/2\ Aa\omega_1$$

$$E(\omega_3 + \tau^2\omega_3^3) = -1/2\ Aa\omega_1 - 1/2\ \tau Ba\omega_1\omega_3$$

$$\boxed{E = \frac{-Aa\omega_1 - \tau Ba\omega_1\omega_3}{2(\omega_3 + \tau^2\omega_3^3)}} \tag{A-26}$$

$$D = \frac{1/2\ Ba\omega_1 + \tau F \omega_4^2}{\omega_4} \tag{A-20a}$$

$$F = \frac{1/2\ Aa\omega_1 - \tau D \omega_4^2}{\omega_4} \tag{A-22a}$$

Substitute (A-22a) into (A-20)

$$D\omega_4 - \tau\,\omega_4^2 \left[\frac{1/2\ Aa\omega_1 - \tau D\omega_4^2}{\omega_4}\right] = \frac{Ba\omega_1}{2} \tag{A-27}$$

$$D\omega_4 - 1/2\ \tau Aa\omega_1\omega_4 + \tau^2 D\omega_4^3 = 1/2\ Ba\omega_1$$

$$D(\omega_4 + \tau^2\omega_4^3) = 1/2\ Ba\omega_1 + 1/2\ \tau Aa\omega_1\omega_4$$

$$\boxed{D = \frac{Ba\omega_1 + \tau Aa\omega_1\omega_4}{2(\omega_4 + \tau^2\omega_4^3)}} \tag{A-28}$$

Substitute (A-20a) into (A-22)

$$-F\omega_4 - \tau\,\omega_4^2 \left[\frac{1/2Ba\omega_1 + \tau F\omega_4^2}{\omega_4}\right] = -1/2\ Aa\omega_1 \tag{A-29}$$

$$-F\omega_4 - 1/2\ \tau Ba\omega_1\omega_4 - \tau^2 F\omega_4^3 = -1/2\ Aa\omega_1$$

$$F(\omega_4 + \tau^2\omega_4^3) = 1/2\ Aa\omega_1 - 1/2\ \tau Ba\omega_1\omega_4$$

$$\boxed{F = \frac{Aa\omega_1 - \tau Ba\omega_1\omega_4}{2(\omega_4 + \tau^2\omega_4^3)}} \tag{A-30}$$

An equation for x_5 is obtained by substituting (A-17) into the following:

$$\dot{x}_5 + \tau\ddot{x}_5 = (a\omega_1 \cos\omega_1)\,x_3 \tag{A-31}$$

$$\dot{x}_5 + \tau\ddot{x}_5 = a\omega_1 \cos\omega_1 t\,[C\sin\omega_3 t + D\sin\omega_4 t + E\cos\omega_3 t + F\cos\omega_4 t] \tag{A-32}$$

$$\tau \ddot{x}_5 + \dot{x}_5 = Ca\omega_1 \cos \omega_1 t \sin \omega_3 t + Da\omega_1 \cos \omega_1 t \sin \omega_4 t + Ea\omega_1 \cos \omega_1 t \cos \omega_3 t$$

$$+ Fa\omega_1 \cos \omega_1 t \cos \omega_4 t \qquad \text{(A-32a)}$$

Trigonometry Relations

$$\cos \omega_1 t \sin \omega_3 t = \frac{1}{2} [\sin(\omega_1 t + \omega_3 t) - \sin(\omega_1 t - \omega_3 t)] \qquad \text{(A-33a)}$$

$$\cos \omega_1 t \sin \omega_4 t = \frac{1}{2} [\sin(\omega_1 t + \omega_4 t) - \sin(\omega_1 t - \omega_4 t)] \qquad \text{(A-33b)}$$

$$\cos \omega_1 t \cos \omega_3 t = \frac{1}{2} [\cos(\omega_1 t + \omega_3 t) + \cos(\omega_1 t - \omega_3 t)] \qquad \text{(A-33c)}$$

$$\cos \omega_1 t \cos \omega_4 t = \frac{1}{2} [\cos(\omega_1 t + \omega_4 t) + \cos(\omega_1 t - \omega_4 t)] \qquad \text{(A-33d)}$$

$$\omega_3 t = (\omega_1 t + \omega_2 t) \qquad \text{(A-34)}$$

$$\omega_4 t = (\omega_1 t - \omega_2 t) \qquad \text{(A-35)}$$

$$\tau \ddot{x}_5 + \dot{x}_5 = \frac{1}{2} Ca\omega_1 [\sin(2\omega_1 t + \omega_2 t) - \sin(-\omega_2 t)]$$

$$+ \frac{1}{2} Da\omega_1 [\sin(2\omega_1 t - \omega_2 t) - \sin(\omega_2 t)]$$

$$+ \frac{1}{2} Ea\omega_1 [\cos(2\omega_1 t + \omega_2 t) + \cos(-\omega_2 t)]$$

$$+ \frac{1}{2} Fa\omega_1 [\cos(2\omega_1 t - \omega_2 t) + \cos(\omega_2 t)] \qquad \text{(A-36)}$$

Let

$$x_5 = G \sin(2\omega_1 t + \omega_2 t) + H_1 \cos(2\omega_1 t + \omega_2 t) + J \sin(2\omega_1 t - \omega_2 t) + K \cos(2\omega_1 t - \omega_2 t)$$

$$+ L \sin(-\omega_2 t) + M \cos(-\omega_2 t) + N \sin(\omega_2 t) + P \cos(\omega_2 t) \qquad \text{(A-37)}$$

$$\dot{x}_5 = G(2\omega_1 + \omega_2) \cos(2\omega_1 t + \omega_2 t) - H_1(2\omega_1 + \omega_2) \sin(2\omega_1 t + \omega_2 t)$$

$$+ J(2\omega_1 - \omega_2) \cos(2\omega_1 t - \omega_2 t) - K(2\omega_1 - \omega_2) \sin(2\omega_1 t - \omega_2 t)$$

$$- L\omega_2 \cos(-\omega_2 t) + M\omega_2 \sin(-\omega_2 t) + N\omega_2 \cos(\omega_2 t) - P\omega_2 \sin(\omega_2 t) \qquad \text{(A-37a)}$$

$$\ddot{x}_5 = -G(2\omega_1+\omega_2)^2\sin(2\omega_1 t+\omega_2 t) - H_1(2\omega_1+\omega_2)^2\cos(2\omega_1 t+\omega_2 t)$$
$$-J(2\omega_1-\omega_2)^2\sin(2\omega_1 t-\omega_2 t) - K(2\omega_1-\omega_2)^2\cos(2\omega_1 t-\omega_2 t)$$
$$-L\omega_2^2\sin(-\omega_2 t) - M\omega_2^2\cos(-\omega_2 t) - N\omega_2^2\sin(\omega_2 t) - P\omega_2^2\cos(\omega_2 t) \tag{A-37b}$$

Substitute (A-37), (A-37a) and (A-37b) into (A-36).

$$-\tau G(2\omega_1+\omega_2)^2\sin(2\omega_1 t+\omega_2 t) - \tau H_1(2\omega_1+\omega_2)^2\cos(2\omega_1 t+\omega_2 t)$$
$$-\tau J(2\omega_1-\omega_2)^2\sin(2\omega_1 t-\omega_2 t) - \tau K(2\omega_1-\omega_2)^2\cos(2\omega_1 t-\omega_2 t)$$
$$-\tau L\omega_2^2\sin(-\omega_2 t) - \tau M\omega_2^2\cos(-\omega_2 t) - \tau N\omega_2^2\sin(\omega_2 t) - \tau P\omega_2^2\cos(\omega_2 t)$$
$$+ G(2\omega_1+\omega_2)\cos(2\omega_1 t+\omega_2 t) - H_1(2\omega_1+\omega_2)\sin(2\omega_1 t+\omega_2 t)$$
$$+ J(2\omega_1-\omega_2)\cos(2\omega_1 t-\omega_2 t) - K(2\omega_1-\omega_2)\sin(2\omega_1 t-\omega_2 t) - L\omega_2\cos(-\omega_2 t)$$
$$+ M\omega_2\sin(-\omega_2 t) + N\omega_2\cos(\omega_2 t) - P\omega_2\sin(\omega_2 t) = 1/2\, Ca\omega_1\sin(2\omega_1+\omega_2 t)$$
$$-1/2\, Ca\omega_1\sin(-\omega_2 t) + 1/2\, Da\omega_1\sin(2\omega_1 t-\omega_2 t) - 1/2\, Da\omega_1\sin(\omega_2 t)$$
$$+1/2\, Ea\omega_1\cos(2\omega_1 t+\omega_2 t) + 1/2\, Ea\omega_1\cos(-\omega_2 t) + 1/2\, Fa\omega_1\cos(2\omega_1 t-\omega_2 t)$$
$$+1/2\, Fa\omega_1\cos(\omega_2 t) \tag{A-38}$$

Now equate like parts and solve for coefficients.

$$\sin(2\omega_1 t+\omega_2 t): \quad -\tau G(2\omega_1+\omega_2)^2 - H_1(2\omega_1+\omega_2) = \frac{1}{2}Ca\omega_1 \tag{A-39}$$

$$\cos(2\omega_1 t+\omega_2 t): \quad -\tau H_1(2\omega_1+\omega_2)^2 + G(2\omega_1+\omega_2) = \frac{1}{2}Ea\omega_1 \tag{A-40}$$

$$\sin(2\omega_1 t-\omega_2 t): \quad -\tau J(2\omega_1-\omega_2)^2 - K(2\omega_1-\omega_2) = \frac{1}{2}Da\omega_1 \tag{A-41}$$

$$\cos(2\omega_1 t-\omega_2 t): \quad -\tau K(2\omega_1-\omega_2)^2 + J(2\omega_1-\omega_2) = \frac{1}{2}Fa\omega_1 \tag{A-42}$$

$$\sin(-\omega_2 t): \quad -\tau L\omega_2^2 + M\omega_2 = -1/2\, Ca\omega_1 \tag{A-43}$$

[illegible]

$$\cos(-\omega_2 t): \qquad -\tau M\omega_2^2 - L\omega_2 = \frac{1}{2} Ea\omega_1 \tag{A-44}$$

$$\sin(\omega_2 t): \qquad -\tau N\omega_2^2 - P\omega_2 = -\frac{1}{2} Da\omega_1 \tag{A45}$$

$$\cos(\omega_2 t): \qquad -\tau P\omega_2^2 + N\omega_2 = \frac{1}{2} Fa\omega_1 \tag{A-46}$$

$$H_1 = \frac{-1/2\, Ca\omega_1 - \tau G(2\omega_1 + \omega_2)^2}{(2\omega_1 + \omega_2)} \tag{A-39a}$$

Substitute (A-39a) into (A-40) and solve for G:

$$-\tau(2\omega_1 + \omega_2)^2 \left[\frac{-1/2\, Ca\omega_1 - \tau G(2\omega_1 + \omega_2)^2}{(2\omega_1 + \omega_2)}\right] + G(2\omega_1 + \omega_2) = \frac{1}{2} Ea\omega_1 \tag{A-40}$$

$$\frac{1}{2}\tau Ca\omega_1(2\omega_1 + \omega_2) + \tau^2 G(2\omega_1 + \omega_2)^3 + G(2\omega_1 + \omega_2) = \frac{1}{2} Ea\omega_1$$

$$G\left[(2\omega_1 + \omega_2) + \tau^2(2\omega_1 + \omega_2)^3\right] = \frac{1}{2} Ea\omega_1 - \frac{1}{2}\tau Ca\omega_1(2\omega_1 + \omega_2)$$

$$G = \frac{Ea\omega_1 - \tau Ca\omega_1(2\omega_1 + \omega_2)}{2[(2\omega_1 + \omega_2) + \tau^2(2\omega_1 + \omega_2)^3]} \tag{A-47}$$

or

$$\boxed{G = \frac{a\omega_1[E - \tau C(2\omega_1 + \omega_2)]}{2[(2\omega_1 + \omega_2) + \tau^2(2\omega_1 + \omega_2)^3]}} \tag{A-47}$$

$$G = \frac{1/2\, Ea\omega_1 + \tau H_1(2\omega_1 + \omega_2)^2}{(2\omega_1 + \omega_2)} \tag{A-40a}$$

Substitute (A-40a) into (A-39) and solve for H_1:

$$-\tau(2\omega_1 + \omega_2)^2 \left[\frac{1/2\, Ea\omega_1 + \tau H_1(2\omega_1 + \omega_2)^2}{(2\omega_1 + \omega_2)}\right] - H_1(2\omega_1 + \omega_2) = \frac{1}{2} Ca\omega_1 \tag{A-39}$$

$$-\frac{1}{2}\tau Ea\omega_1(2\omega_1+\omega_2) - \tau H_1(2\omega_1+\omega_2)^3 - H_1(2\omega_1+\omega_2) = \frac{1}{2}Ca\omega_1$$

$$H_1[(2\omega_1+\omega_2) + \tau^2(2\omega_1+\omega_2)^3] = -\frac{1}{2}Ca\omega_1 - \frac{1}{2}\tau Ea\omega_1(2\omega_1+\omega_2)$$

$$H_1 = \frac{-Ca\omega_1 - \tau Ea\omega_1(2\omega_1+\omega_2)}{2[(2\omega_1+\omega_2) + \tau^2(2\omega_1+\omega_2)^3} \qquad \text{(A-48)}$$

or

$$\boxed{H_1 = \frac{a\omega_1[-C - \tau E(2\omega_1+\omega_2)]}{2[(2\omega_1+\omega_2) + \tau^2(2\omega_1+\omega_2)^3]}} \qquad \text{(A-48)}$$

$$J = \frac{1/2\, Fa\omega_1 + \tau K(2\omega_1-\omega_2)^2}{(2\omega_1-\omega_2)} \qquad \text{(A-42a)}$$

Substitute (A-42a) into (A-41) and solve for K:

$$-\tau(2\omega_1-\omega_2)^2\left[\frac{1/2\, Fa\omega_1 + \tau K(2\omega_1-\omega_2)^2}{(2\omega_1-\omega_2)}\right] - K(2\omega_1-\omega_2) = \frac{1}{2}Da\omega_1 \qquad \text{(A-41)}$$

$$-\frac{1}{2}\tau Fa\omega_1(2\omega_1-\omega_2) - \tau^2 K(2\omega_1-\omega_2)^3 - K(2\omega_1-\omega_2) = \frac{1}{2}Da\omega_1$$

$$K[(2\omega_1-\omega_2) + \tau^2(2\omega_1-\omega_2)^3] = -\frac{1}{2}Da\omega_1 - \frac{1}{2}\tau Fa\omega_1(2\omega_1-\omega_2)$$

$$\boxed{K = \frac{a\omega_1[-D - \tau F(2\omega_1-\omega_2)]}{2[(2\omega_1-\omega_2) + \tau^2(2\omega_1-\omega_2)^3]}} \qquad \text{(A-49)}$$

$$K = \frac{-1/2\, Da\omega_1 - \tau J(2\omega_1-\omega_2)^2}{(2\omega_1-\omega_2)} \qquad \text{(A-41a)}$$

Substitute (A-41a) into (A-42) and solve for J :

$$-\tau(2\omega_1-\omega_2)^2\left[\frac{-1/2\,Da\omega_1-\tau J(2\omega_1-\omega_2)^2}{(2\omega_1-\omega_2)}\right]+J(2\omega_1-\omega_2)=\frac{1}{2}Fa\omega_1 \tag{A-42}$$

$$\frac{1}{2}\tau Da\omega_1(2\omega_1-\omega_2)+\tau^2 J(2\omega_1-\omega_2)^3+J(2\omega_1-\omega_2)=\frac{1}{2}Fa\omega_1$$

$$J[(2\omega_1-\omega_2)+\tau^2(2\omega_1-\omega_2)^3]=\frac{1}{2}Fa\omega_1-\frac{1}{2}\tau Da\omega_1(2\omega_1-\omega_2)$$

$$\boxed{J=\frac{a\omega_1[F-\tau D(2\omega_1-\omega_2)]}{2[(2\omega_1-\omega_2)+\tau^2(2\omega_1-\omega_2)^3]}} \tag{A-50}$$

$$M=\frac{-1/2\,Ca\omega_1+\tau L\omega_2^2}{\omega_2} \tag{A-43a}$$

Substitute (A-43a) into (A-44) and solve for L :

$$-\tau\omega_2^2\left[\frac{-1/2\,Ca\omega_1+\tau L\omega_2^2}{\omega_2}\right]-L\omega_2=\frac{1}{2}Ea\omega_1 \tag{A-44}$$

$$\frac{1}{2}\tau Ca\omega_1\omega_2-\tau^2 L\omega_2^3-L\omega_2=\frac{1}{2}Ea\omega_1$$

$$L(\omega_2+\tau^2\omega_2^3)=-\frac{1}{2}Ea\omega_1+\frac{1}{2}\tau Ca\omega_1\omega_2$$

$$\boxed{L=\frac{a\omega_1[-E+\tau C\omega_2]}{2[\omega_2+\tau^2\omega_2^3]}} \tag{A-51}$$

$$L=\frac{-1/2\,Ea\omega_1-\tau M\omega_2^2}{\omega_2} \tag{A-44a}$$

Substitute (A-44a) into (A-43) and solve for M :

$$-\tau\,\omega_2^2\left[\frac{-1/2\,Ea\omega_1 - \tau\,M\,\omega_2^2}{\omega_2}\right] + M\omega_2 = -\frac{1}{2}\,Ca\omega_1 \qquad \text{(A-43)}$$

$$\frac{1}{2}\,\tau\,Ea\omega_1\omega_2 + \tau^2 M\,\omega_2^3 + M\omega_2 = -\frac{1}{2}\,Ca\omega_1$$

$$M(\omega_2 + \tau^2\,\omega_2^3) = -\frac{1}{2}\,Ca\omega_1 - \frac{1}{2}\,\tau\,Ea\omega_1\omega_2$$

$$\boxed{M = \frac{a\omega_1\,(-C - \tau\,E\omega_2)}{2(\omega_2 + \tau^2\,\omega_2^3)}} \qquad \text{(A-52)}$$

$$P = \frac{1/2\,Da\omega_1 - \tau N\,\omega_2^2}{\omega_2} \qquad \text{(A-45a)}$$

Substitute (A-45a) into (A-46) and solve for N:

$$-\tau\,\omega_2^2\left[\frac{1/2\,Da\omega_1 - \tau N\,\omega_2^2}{\omega_2}\right] + N\omega_2 = \frac{1}{2}\,Fa\omega_1 \qquad \text{(A-46)}$$

$$-\frac{1}{2}\,\tau\,Da\omega_1\omega_2 + \tau^2 N\,\omega_2^3 + N\omega_2 = \frac{1}{2}\,Fa\omega_1$$

$$N(\omega_2 + \tau^2\,\omega_2^3) = \frac{1}{2}\,Fa\omega_1 + \frac{1}{2}\,\tau\,Da\omega_1\omega_2$$

$$\boxed{N = \frac{a\omega_1\,(F + \tau D\omega_2)}{2(\omega_2 + \tau^2\,\omega_2^3)}} \qquad \text{(A-53)}$$

$$N = \frac{1/2\,Fa\omega_1 + \tau\,P\,\omega_2^2}{\omega_2} \qquad \text{(A-46a)}$$

Substitute (A-46a) into (A-45) and solve for P :

$$- \tau \omega_2^2 \left[\frac{1/2 \, Fa\omega_1 + \tau P \omega_2^2}{\omega_2} \right] - P\omega_2 = - \frac{1}{2} Da\omega_1 \qquad \text{(A-45)}$$

$$- \frac{1}{2} \tau Fa\omega_1 \omega_2 - \tau^2 P \omega_2^3 - P\omega_2 = - \frac{1}{2} Da\omega_1$$

$$P(\omega_2 + \tau^2 \omega_2^3) = \frac{1}{2} Da\omega_1 - \frac{1}{2} \tau Fa\omega_1 \omega_2$$

$$\boxed{P = \frac{a\omega_1 (D - \tau F\omega_2)}{2(\omega_2 + \tau^2 \omega_2^3)}} \qquad \text{(A-54)}$$

$$x = x_1 + x_3 + x_5 + \dots \; x_{2m-1} + x_{2m} \qquad \text{(A-3)}$$

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \qquad \text{(A-7)}$$

$$x_3 = C \sin \omega_3 t + D \sin \omega_4 t + E \cos \omega_3 t + F \cos \omega_4 t \qquad \text{(A-17)}$$

$$x_5 = G \sin(2\omega_1 t + \omega_2 t) + H_1 \cos(2\omega_1 t + \omega_2 t) + J \sin(2\omega_1 t - \omega_2 t) + K \cos(2\omega_1 t - \omega_2$$

$$+ L \sin(-\omega_2 t) + M \cos(-\omega_2 t) + N \sin(\omega_2 t) + P \cos(\omega_2 t) \qquad \text{(A-37)}$$

Terms beyond x_5 may be neglected if the order of magnitude of a is much smaller than unity (maximum a on the MIT angular shaker is 2×10^{-3} radian). If x_1 is defined to be of the order of magnitude of one, x_3 is of the order of magnitude of a, and x_5 is of the order of magnitude of a^2. Hence each successive term in the series of x is smaller than its predecessor in accordance with the value of a.

For further discussion on convergence and a process for iteration, ref. 10 is recommended.

If only the terms associated with x_1, x_3, and x_5 are considered, it is readily apparent that the time average of drift for the sinusoids is zero. Close scrutiny, however, shows drift possibilities as ω_1 and ω_2 approach one another in value and also when harmonics are present.

The specific case where $\omega_1 = \omega_2$ has been investigated in ref. 10. The general case which is the subject of this chapter degenerates to the specific case when $\omega_1 = \omega_2$. Under such circumstances $D \sin \omega_4 t$ and $F \cos \omega_4 t$ require L' Hospital's rule for proper evaluation, but eventually yield a drift term which is a direct function of time.

The other case of interest concerns harmonics. The series terms associated with x_5 require L' Hospital's rule for evaluation when $2\omega_1 = \omega_2$. Under such a condition drift terms appear which are direct functions of time.

An example of drift caused by the existence of harmonics such as when $2\omega_1 = \omega_2$ is shown for the term $J \sin(2\omega_1 t - \omega_2 t)$.

If $2\omega_1 \rightarrow \omega_2$, then $(2\omega_1 - \omega_2) \rightarrow 0$ (A-55)

$$J \sin(2\omega_1 t - \omega_2 t) = \frac{a\omega_1 [F - \tau D(2\omega_1 - \omega_2)]}{2(2\omega_1 - \omega_2)[1 + \tau^2 (2\omega_1 - \omega_2)^2]} \sin(2\omega_1 t - \omega_2 t) \qquad \text{(A-56)}$$

Let $(2\omega_1 t - \omega_2 t) = (2\omega_1 - \omega_2) t = yt$ (A-57)

$$\lim_{y \rightarrow 0} J \sin yt = \frac{0}{0} \qquad \text{apply L' Hospital's Rule} \qquad \text{(A-58)}$$

$$\frac{\frac{d}{dy}[a\omega_1 (F - \tau D y) \sin yt]}{\frac{d}{dy}[2y(1 + \tau^2 y^2)]} = \frac{a\omega_1}{2} \left[\frac{-\tau D \sin yt + (F - \tau D y)(t \cos yt)}{(1 + \tau^2 y^2) + y(2\tau^2 y)} \right] \qquad \text{(A-59)}$$

$$\lim_{y \to 0} J \sin yt = \frac{a\omega_1}{2}\left[\frac{(F)(t)}{1}\right] = \frac{Fa\omega_1 t}{2} \tag{A-60}$$

If terms for x_7, x_9, etc. were evaluated, drift terms would appear for higher harmonics. As the series has an infinite number of terms, it seems apparent that any two given input frequencies might create some type of harmonic drift, but the magnitude of "a" associated with such drift may make the consideration of the value a folly when compared to the manufacturing tolerances used when constructing gyros.

The case of negligible elastic restraint shows the possibility of drift terms. However, for such a gyroscope, the shortened series representing drift is valid only when values of a, b, and c are small compared to unity. Furthermore, man can only produce a gyro which is not perfect. Hence terms in the series which are of less consequence than the manufacturing tolerances will have only a theoretical importance and should not enter practical considerations.

A. 2 Series Solution for Finite Elastic Restraint Between the Float and Case About the Output Axis ($k \neq 0$)

The general equation is :

$$(k - a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \tag{A-1}$$

In accordance with the series established in Section 5. 6, the interest lies with the following terms :

$$x = x_1 + x_3 + x_5 + \dots \, x_{2m-1} + x_{2m} \tag{A-3}$$

where:

$$kx_1 + \dot{x}_1 + \tau \ddot{x}_1 = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \tag{A-61}$$

$$kx_3 + \dot{x}_3 + \tau \ddot{x}_3 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad \text{(A-62)}$$

$$kx_5 + \dot{x}_5 + \tau \ddot{x}_5 = (a\omega_1 \cos \omega_1 t)\, x_3 \qquad \text{(A-63)}$$

$$(k - a\omega_1 \cos \omega_1 t)\, x + \dot{x} + \tau \ddot{x} = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \qquad \text{(A-1)}$$

Let:

$$x = x_1 + x_2 \qquad \text{(A-64)}$$

where:

$$kx_1 + \dot{x}_1 + \tau \ddot{x}_1 = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \qquad \text{(A-65)}$$

Substitute (A-64) and (A-65) into (A-1).

$$(k - a\omega_1 \cos \omega_1 t)(x_1 + x_2) + (\dot{x}_1 + \dot{x}_2) + \tau (\ddot{x}_1 + \ddot{x}_2) = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \qquad \text{(A-66)}$$

or

$$(k - a\omega_1 \cos \omega_1 t)\, x_2 + \dot{x}_2 + \tau \ddot{x}_2 = (a\omega_1 \cos \omega_1 t)\, x_1 \qquad \text{(A-66a)}$$

Try a solution for (A-64) of:

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \qquad \text{(A-67)}$$

$$\dot{x}_1 = A\omega_2 \cos \omega_2 t - B\omega_2 \sin \omega_2 t \qquad \text{(A-67a)}$$

$$\ddot{x}_1 = A\omega_2^2 \sin \omega_2 t - B\omega_2^2 \cos \omega_2 t \qquad \text{(A-67b)}$$

Substitute (A-67a) and (A-67b) into (A-65).

$$kA \sin \omega_2 t + kB \cos \omega_2 t + A\omega_2 \cos \omega_2 t - B\omega_2 \sin \omega_2 t - \tau A\omega_2^2 \sin \omega_2 t - \tau B\omega_2^2 \cos \omega_2 t = b\omega_2 \cos \omega_2 t - c\omega_2 \sin \omega_2 t + kb \sin \omega_2 t + kc \cos \omega_2 t \qquad \text{(A-65)}$$

Equate coefficients of like terms.

$$\sin \omega_2 t: \quad kA - B\omega_2 - \tau A\omega_2^2 = -c\omega_2 + kb \qquad \text{(A-68)}$$

$$\cos \omega_2 t: \quad kB + A\omega_2 - \tau B\omega_2^2 = b\omega_2 + kc \qquad \text{(A-69)}$$

$$A(k - \tau\omega_2^2) = B\omega_2 - c\omega_2 + kb \tag{A-68a}$$

$$A = \frac{\tau B\omega_2^2 - kB + b\omega_2 + kc}{\omega_2} = \tau B\omega_2 - \frac{kB}{\omega_2} + b + \frac{kc}{\omega_2} \tag{A-69a}$$

Substitute (A-69a) into (A-68a) and multiply through by ω_2:

$$k\tau B\omega_2^2 - k^2 B + kb\omega_2 + k^2 C - \tau^2 B\omega_2^4 + \tau kB\omega_2^2 - \tau b\omega_2^3 - \tau kc\,\omega_2^2$$
$$= B\omega_2^2 - c\omega_2^2 + kb\,\omega_2 \tag{A-70}$$

$$B[\ \tau\omega_2^2 - k^2 - \tau^2\omega_2^4 + \tau k\omega_2^2 - \omega_2^2] = \tau b\omega_2^3 - k^2 c + \tau kc\,\omega_2^2 - c\omega_2^2$$

$$B = \frac{\tau b\omega_2 - c\left(\frac{k}{\omega_2}\right)^2 + \tau kc - c}{2k\tau - \left(\frac{k}{\omega_2}\right)^2 - \tau^2\omega_2^2 - 1} = \frac{c\left[1 + \left(\frac{k}{\omega_2}\right)^2\right] - \tau\omega_2 c\left[\frac{b}{c} + \frac{k}{\omega_2}\right]}{1 + \left[\left(\frac{k}{\omega_2}\right) - \tau\omega_2\right]^2}$$

$$\boxed{B = \frac{c\left[1 + \frac{k}{\omega_2}\right]^2 - \tau\omega_2 c\left[\frac{b}{c} + \frac{k}{\omega_2}\right]}{1 + \left[\left(\frac{k}{\omega_2}\right) - \tau\omega_2\right]^2}} \tag{A-71}$$

$$B = \frac{Ak - A\tau\omega_2^2 + c\omega_2 - kb}{\omega_2} = \frac{Ak}{\omega_2} - A\tau\omega_2 + c - \frac{kb}{\omega_2} \tag{A-68b}$$

Substitute (A-68b) into (A-69a) and solve for A:

$$A = \frac{(\tau\omega_2^2 - k)}{\omega_2}\left[\frac{Ak}{\omega_2} - A\tau\omega_2 + c - \frac{kb}{\omega_2}\right] + b + \frac{kc}{\omega_2} \tag{A-69}$$

$$A = \tau Ak - A\tau^2\omega_2^2 + \tau c\omega_2 - \tau kb - \frac{Ak^2}{\omega_2^2} + Ak\tau - \frac{kc}{\omega_2} + \frac{k^2 b}{\omega_2^2} + b + \frac{kc}{\omega_2}$$

$$A\left[1 - \tau k + \tau^2\omega_2^2 + \left(\frac{k}{\omega_2}\right)^2 - \tau k\right] = \tau c\omega_2 - \tau kb + \frac{k^2 b}{\omega_2^2} + b$$

$$\boxed{A = \frac{b\left[1 + \left(\frac{k}{\omega_2}\right)^2\right] + \tau\omega_2 c\left[1 - \frac{b}{c}\left(\frac{k}{\omega_2}\right)\right]}{1 + \left[\left(\frac{k}{\omega_2}\right) - \tau\omega_2\right]^2}} \tag{A-72}$$

$$(k - a\omega_1 \cos\omega_1 t)\, x_2 + \dot{x}_2 + \tau\ddot{x}_2 = (a\omega_1 \cos\omega_1 t)\, x_1 \tag{A-66a}$$

Let:

$$x_2 = x_3 + x_4 \tag{A-73}$$

where:

$$kx_3 + \dot{x}_3 + \tau\ddot{x}_3 = (a\omega_1 \cos\omega_1 t)\, x_1 \tag{A-74}$$

Substitute (A-73) and (A-74) into (A-66a).

$$(k - a\omega_1 \cos\omega_1 t)(x_3 + x_4) + (\dot{x}_3 + \dot{x}_4) + \tau(\ddot{x}_3 + \ddot{x}_4) = (a\omega_1 \cos\omega_1 t)\, x_1 \tag{A-66a}$$

or

$$(k - a\omega_1 \cos\omega_1 t)\, x_4 + \dot{x}_4 + \tau\ddot{x}_4 = (a\omega_1 \cos\omega_1 t)\, x_3 \tag{A-66a}$$

A solution is desired for the following equation:

$$kx_3 + \dot{x}_3 + \tau\ddot{x}_3 = a\omega_1 A\cos\omega_1 t\sin\omega_2 t + a\omega_1 B\cos\omega_1 t\cos\omega_2 t \tag{A-74a}$$

Let:

$$\begin{aligned} x_3 &= D\sin\omega_1 t\sin\omega_2 t + E\sin\omega_1 t\cos\omega_2 t \\ &\quad + F\cos\omega_1 t\sin\omega_2 t + G\cos\omega_1 t\cos\omega_2 t \end{aligned} \tag{A-75}$$

$$
\begin{aligned}
\dot{x}_3 = {} & D\omega_1 \cos\omega_1 t \sin\omega_2 t + D\omega_2 \sin\omega_1 t \cos\omega_2 t \\
& + E\omega_1 \cos\omega_1 t \cos\omega_2 t - E\omega_2 \sin\omega_1 t \sin\omega_2 t \\
& - F\omega_1 \sin\omega_1 t \sin\omega_2 t + F\omega_2 \cos\omega_1 t \cos\omega_2 t \\
& - G\omega_1 \sin\omega_1 t \cos\omega_2 t - G\omega_2 \cos\omega_1 t \sin\omega_2 t
\end{aligned}
\tag{A-75a}
$$

$$
\begin{aligned}
\ddot{x}_3 = {} & -D\omega_1^2 \sin\omega_1 t \sin\omega_2 t + D\omega_1\omega_2 \cos\omega_1 t \cos\omega_2 t \\
& + D\omega_1\omega_2 \cos\omega_1 t \cos\omega_2 t - D\omega_2^2 \sin\omega_1 t \sin\omega_2 t \\
& - E\omega_1^2 \sin\omega_1 t \cos\omega_2 t - E\omega_1\omega_2 \cos\omega_1 t \sin\omega_2 t \\
& - E\omega_1\omega_2 \cos\omega_1 t \sin\omega_2 t - E\omega_2^2 \sin\omega_1 t \cos\omega_2 t \\
& - F\omega_1^2 \cos\omega_1 t \sin\omega_2 t - F\omega_1\omega_2 \sin\omega_1 t \cos\omega_2 t \\
& - F\omega_1\omega_2 \sin\omega_1 t \cos\omega_2 t - F\omega_2^2 \cos\omega_1 t \sin\omega_2 t \\
& - G\omega_1^2 \cos\omega_1 t \cos\omega_2 t + G\omega_1\omega_2 \sin\omega_1 t \sin\omega_2 t \\
& + G\omega_1\omega_2 \sin\omega_1 t \sin\omega_2 t - G\omega_2^2 \cos\omega_1 t \cos\omega_2 t
\end{aligned}
\tag{A-75b}
$$

Now substitute (A-75), (A-75a) and (A-75b) into (A-74a) and solve for D, E, F, and G.

$$
\begin{aligned}
& (kD \sin\omega_1 t \sin\omega_2 t + kE \sin\omega_1 t \cos\omega_2 t + kF \cos\omega_1 t \sin\omega_2 t + kG \cos\omega_1 t \cos\omega_2 t) \\
& + (D\omega_1 \cos\omega_1 t \sin\omega_2 t + D\omega_2 \sin\omega_1 t \cos\omega_2 t + E\omega_1 \cos\omega_1 t \cos\omega_2 t - E\omega_2 \sin\omega_1 t \sin\omega_2 t \\
& \quad - F\omega_1 \sin\omega_1 t \sin\omega_2 t + F\omega_2 \cos\omega_1 t \cos\omega_2 t - G\omega_1 \sin\omega_1 t \cos\omega_2 t - G\omega_2 \cos\omega_1 t \sin\omega_2 t) \\
& + (-\tau D\omega_1^2 \sin\omega_1 t \sin\omega_2 t + \tau D\omega_1\omega_2 \cos\omega_1 t \cos\omega_2 t + \tau D\omega_1\omega_2 \cos\omega_1 t \cos\omega_2 t \\
& \quad - \tau D\omega_2^2 \sin\omega_1 t \sin\omega_2 t - \tau E\omega_1^2 \sin\omega_1 t \cos\omega_2 t - \tau E\omega_1\omega_2 \cos\omega_1 t \sin\omega_2 t \\
& \quad - \tau E\omega_1\omega_2 \cos\omega_1 t \sin\omega_2 t - \tau E\omega_2^2 \sin\omega_1 t \cos\omega_2 t - \tau F\omega_1^2 \cos\omega_1 t \sin\omega_2 t \\
& \quad - \tau F\omega_1\omega_2 \sin\omega_1 t \cos\omega_2 t - \tau F\omega_1\omega_2 \sin\omega_1 t \cos\omega_2 t - \tau F\omega_2^2 \cos\omega_1 t \sin\omega_2 t \\
& \quad - \tau G\omega_1^2 \cos\omega_1 t \cos\omega_2 t + \tau G\omega_1\omega_2 \sin\omega_1 t \sin\omega_2 t + \tau G\omega_1\omega_2 \sin\omega_1 t \sin\omega_2 t \\
& \quad - \tau G\omega_2^2 \cos\omega_1 t \cos\omega_2 t) = Aa\omega_1 \cos\omega_1 t \sin\omega_2 t + Ba\omega_1 \cos\omega_1 t \cos\omega_2 t
\end{aligned}
\tag{A-76}
$$

Now equate like terms of the various trigonometric function combinations and solve for D, E, F, and G. (Note: A and B are known)

$(\sin \omega_1 t \sin \omega_2 t)$

$$kD - E\omega_2 - F\omega_1 - \tau D \omega_1^2 - \tau D \omega_2^2 + \tau G \omega_1 \omega_2 + \tau G \omega_1 \omega_2 = 0$$

$$kD - E\omega_2 - F\omega_1 - D(\tau \omega_1^2 + \tau \omega_2^2) + 2\tau G \omega_1 \omega_2 = 0$$

$$D(k - \tau \omega_1^2 - \tau \omega_2^2) - E\omega_2 - F\omega_1 + G(2\tau \omega_1 \omega_2) = 0 \tag{A-77}$$

$(\sin \omega_1 t \cos \omega_2 t)$

$$kE + D\omega_2 - G\omega_1 - \tau E \omega_1^2 - \tau E \omega_2^2 - \tau F \omega_1 \omega_2 - \tau F \omega_1 \omega_2 = 0$$

$$D\omega_2 + E(k - \tau \omega_1^2 - \tau \omega_2^2) - F(2\tau \omega_1 \omega_2) - G\omega_1 = 0 \tag{A-78}$$

$(\cos \omega_1 t \sin \omega_2 t)$

$$kF + D\omega_1 - G\omega_2 - \tau E \omega_1 \omega_2 - \tau E \omega_1 \omega_2 - \tau F \omega_1^2 - \tau F \omega_2^2 = Aa\omega_1$$

$$D\omega_1 - E(2\tau \omega_1 \omega_2) + F(k - \tau \omega_1^2 - \tau \omega_2^2) - G\omega_2 = Aa\omega_1 \tag{A-79}$$

$(\cos \omega_1 t \cos \omega_2 t)$

$$kG + E\omega_1 + F\omega_2 + \tau D \omega_1 \omega_2 + \tau D \omega_1 \omega_2 - \tau G \omega_1^2 - \tau G \omega_2^2 = Ba\omega_1$$

$$D(2\tau \omega_1 \omega_2) + E\omega_1 + F\omega_2 + G(k - \tau \omega_1^2 - \tau \omega_2^2) = Ba\omega_1 \tag{A-80}$$

$$D(k - \tau \omega_1^2 - \tau \omega_2^2) - E\omega_2 - F\omega_1 + G(2\tau \omega_1 \omega_2) = 0 \tag{A-77}$$

$$D\omega_2 + E(k - \tau \omega_1^2 - \tau \omega_2^2) - F(2\tau \omega_1 \omega_2) - G\omega_1 = 0 \tag{A-78}$$

$$D\omega_1 + E(2\tau \omega_1 \omega_2) + F(k - \tau \omega_1^2 - \tau \omega_2^2) - G\omega_2 = Aa\omega_1 \tag{A-79}$$

$$D(2\tau \omega_1 \omega_2) + E\omega_1 + F\omega_2 + G(k - \tau \omega_1^2) = Ba\omega_1 \tag{A-80}$$

Solve for D, E, F, and G by determinants.

$$D = \frac{\begin{vmatrix} 0 & -\omega_2 & -\omega_1 & 2\tau\omega_1\omega_2 \\ 0 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -2\tau\omega_1\omega_2 & -\omega_1 \\ Aa\omega_1 & 2\tau\omega_1\omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ Ba\omega_1 & \omega_1 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\begin{vmatrix} (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 & -\omega_1 & 2\tau\omega_1\omega_2 \\ \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -2\tau\omega_1\omega_2 & -\omega_1 \\ \omega_1 & 2\tau\omega_1\omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ 2\tau\omega_1\omega_2 & \omega_1 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}} \quad \text{(A-81)}$$

(A-82)

Let the denominator of the determinant be known as Δ or (A-82) .

$$\Delta = (k - \tau\omega_1^2 - \tau\omega_2^2) \begin{vmatrix} (k - \tau\omega_1^2 - \tau\omega_2^2) & -2\tau\omega_1\omega_2 & -\omega_1 \\ 2\tau\omega_1\omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ \omega_1 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$+ \omega_2 \begin{vmatrix} \omega_2 & -2\tau\omega_1\omega_2 & -\omega_1 \\ \omega_1 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ 2\tau\omega_1\omega_2 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$- \omega_1 \begin{vmatrix} \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_1 \\ \omega_1 & 2\tau\omega_1\omega_2 & -\omega_2 \\ 2\tau\omega_1\omega_2 & \omega_1 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$-2\tau\omega_1\omega_2 \begin{vmatrix} \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -2\tau\omega_1\omega_2 \\ \omega_1 & 2\tau\omega_1\omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \\ 2\tau\omega_1\omega_2 & \omega_1 & \omega_2 \end{vmatrix}$$

(A-82)

$$
\begin{aligned}
\Delta = {} & (k - \tau\omega_1^2 - \tau\omega_2^2)\Big[(k - \tau\omega_1^2 - \tau\omega_2^2)^3 + \omega_1^2(k - \tau\omega_1^2 - \tau\omega_2^2) - 2\tau\omega_1^2\omega_2^2 \\
& + \omega_2^2(k - \tau\omega_1^2 - \tau\omega_2^2) + 2\tau\omega_1^2\omega_2^2 + 4\tau^2\omega_1^2\omega_2^2(k - \tau\omega_1^2 - \tau\omega_2^2)\Big] \\
& + \omega_2\Big[\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2)^2 + \omega_2^3 + 2\tau\omega_1^2\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2) \\
& + 4\tau^2\omega_1^2\omega_2^3 - \omega_1^2\omega_2 + 2\tau\omega_1^2\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2)\Big] \\
& + (-\omega_1)\Big[2\tau\omega_1\omega_2^2(k - \tau\omega_1^2 - \tau\omega_2^2) + \omega_1\omega_2^2 - \omega_1(k - \tau\omega_1^2 - \tau\omega_2^2)^2 \\
& - 2\tau\omega_1\omega_2^2(k - \tau\omega_1^2 - \tau\omega_2^2) - \omega_1^3 + 4\tau^2\omega_1^3\omega_2^2\Big] \\
& + (-2\tau\omega_1\omega_2)\Big[2\tau\omega_1\omega_2^3 - \omega_1\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2) - \omega_1\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2) \\
& + 2\tau\omega_1\omega_2(k - \tau\omega_1^2 - \tau\omega_2^2)^2 - 2\tau\omega_1^3\omega_2 + 8\tau^3\omega_1^3\omega_2^3\Big]
\end{aligned}
\qquad \text{(A-82)}
$$

$$
\begin{aligned}
\Delta = {} & (k - \tau\omega_1^2 - \tau\omega_2^2)^4 + (2\omega_1^2 + 2\omega_2^2)(k - \tau\omega_1^2 - \tau\omega_2^2)^2 \\
& + 8\tau\omega_1^2\omega_2^2(k - \tau\omega_1^2 - \tau\omega_2^2) - 2\omega_1^2\omega_2^2 + \omega_1^4 + \omega_2^4 - 16\tau^2\omega_1^4\omega_2^4
\end{aligned}
\qquad \text{(A-82a)}
$$

Determinant Form of Coefficients D, E, F, and G.

$$
D = \frac{\begin{vmatrix} 0 & -\omega_2 & -\omega_1 & 2\tau\omega_1\omega_2 \\ 0 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -2\tau\omega_1\omega_2 & -\omega_1 \\ Aa\omega_1 & 2\tau\omega_1\omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ Ba\omega_1 & \omega_1 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\Delta}
\qquad \text{(A-81)}
$$

$$
E = \frac{\begin{vmatrix} (k - \tau\omega_1^2 - \tau\omega_2^2) & 0 & -\omega_1 & 2\tau\omega_1\omega_2 \\ \omega_2 & 0 & -2\tau\omega_1\omega_2 & -\omega_1 \\ \omega_1 & Aa\omega_1 & (k - \tau\omega_1^2 - \tau\omega_2^2) & -\omega_2 \\ 2\tau\omega_1\omega_2 & Ba\omega_1 & \omega_2 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\Delta}
\qquad \text{(A-83)}
$$

$$F = \frac{\begin{vmatrix} (k - \tau\,\omega_1^2 - \tau\,\omega_2^2) & -\omega_2 & 0 & 2\,\tau\,\omega_1\omega_2 \\ \omega_2 & (k - \tau\,\omega_1^2 - \tau\,\omega_2^2) & 0 & -\omega_1 \\ \omega_1 & 2\,\tau\,\omega_1\omega_2 & Aa\omega_1 & -\omega_2 \\ 2\,\tau\,\omega_1\omega_2 & \omega_1 & Ba\omega_1 & (k - \tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\Delta} \tag{A-84}$$

$$G = \frac{\begin{vmatrix} (k - \tau\,\omega_1^2 - \tau\,\omega_2^2) & -\omega_2 & -\omega_1 & 0 \\ \omega_2 & (k - \tau\,\omega_1^2 - \tau\,\omega_2^2) & -2\,\tau\,\omega_1\omega_2 & 0 \\ \omega_1 & 2\;\;\tau\omega_1\omega_2 & (k - \tau\,\omega_1^2 - \tau\omega_2^2) & Aa\omega_1 \\ 2\,\tau\omega_1\omega_2 & \omega_1 & \omega_2 & Ba\omega_1 \end{vmatrix}}{\Delta} \tag{A-85}$$

$$\begin{aligned} \Delta D = Aa\omega_1 & \Big[2\,\tau\omega_1\omega_2^2(k - \tau\omega_1^2 - \tau\,\omega_2^2) + 4\,\tau^2\,\omega_1^3\,\omega_2^2 + \,\omega_1^3 - \omega_1\omega_2^2 \\ & + 2\tau\,\omega_1\omega_2^2\,(k - \tau\omega_1^2 - \tau\,\omega_2^2) + \omega_1\,(k - \tau\,\omega_1^2 - \tau\,\omega_2^2)^2\Big] \\ -Ba\omega_1 & \Big[-2\tau\,\omega_1\omega_2^3 + 2\,\tau\,\omega_2^3\,\omega_2 + 2\tau\,\omega_1\omega_2\,(k - \tau\;\omega_1^2 - \tau\omega_2^2)^2 \\ & + 8\,\tau\,\omega_1^3\,\omega_2^3 - \omega_1\omega_2\,(k - \tau\,\omega_1^2 - \tau\,\omega_2^2) - \omega_1\omega_2\,(k - \tau\,\omega_1^2 - \tau\,\omega_2^2)\Big] \end{aligned} \tag{A-81}$$

$$\begin{aligned} \Delta D = Aa\omega_1 & \Big[\,4\,\tau\,\omega_1\omega_2^2\,(k - \tau\,\omega_1^2 - \tau\,\omega_2^2) + 4\,\tau^2\,\omega_1^3\omega_2^2 + \omega_1^3 - \omega_1\omega_2^2 \\ & + \omega_1\,(k - \tau\,\omega_1^2 - \tau\,\omega_2^2)^2\Big] + Ba\omega_1\Big[2\,\tau\,\omega_1\omega_2^3 - 2\,\tau\,\omega_1^3\,\omega_2 \\ & - 2\,\tau\,\omega_1\omega_2\,(k - \tau\,\omega_1^2 - \tau\omega_2^2)^2 - 8\,\tau\,\omega_1^3\,\omega_2^3 + 2\omega_1\omega_2(k - \tau\,\omega_1^2 - \tau\,\omega_2^2)\Big] \end{aligned} \tag{A-81}$$

$$\Delta E = -Aa\omega_1 \Big[-2\tau\omega_1\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) + 2\tau\omega_1^3\omega_2 + 2\tau\omega_1\omega_2^3$$
$$+ 8\tau^3\omega_1^3\omega_2^3 + \omega_1\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_1\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) \Big]$$
$$+ Ba\omega_1 \Big[2\tau\omega_1\omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_1^3 + 2\tau\omega_1\omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2)$$
$$+ 4\tau^2\omega_1^3\omega_2^2 + \omega_1(k-\tau\omega_1^2-\tau\omega_2^2)^2 + \omega_1\omega_2^2 \Big] \qquad \text{(A-83)}$$

$$\Delta E = Aa\omega_1 \Big[2\tau\omega_1\omega_2(k-\tau\omega_1^2-\tau\omega_2^2)^2 - 2\tau\omega_1^3\omega_2 - 2\tau\omega_1\omega_2^3 - 8\tau^3\omega_1^3\omega_2^3$$
$$- 2\omega_1\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) \Big] + Ba\omega_1 \Big[4\tau\omega_1\omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_1^3$$
$$+ 4\tau^2\omega_1^3\omega_2^2 + \omega_1(k-\tau\omega_1^2-\tau\omega_2^2)^2 + \omega_1\omega_2^2 \Big] \qquad \text{(A-83)}$$

$$\Delta F = Aa\omega_1 \Big[(k-\tau\omega_1^2-\tau\omega_2^2)^3 + 2\tau\omega_1^2\omega_2^2 + 2\tau\omega_1^2\omega_2^2$$
$$- 4\tau^2\omega_1^2\omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_1^2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) \Big]$$
$$- Ba\omega_1 \Big[-\omega_2(k-\tau\omega_1^2-\tau\omega_2^2)^2 + \omega_1^2\omega_2 + 4\tau^2\omega_1^2\omega_2^3$$
$$- 2\tau\omega_1^2\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) + 2\tau\omega_1^2\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) - \omega_2^3 \Big] \qquad \text{(A-84)}$$

$$\Delta F = Aa\omega_1 \Big[(k-\tau\omega_1^2-\tau\omega_2^2)^3 + 4\tau\omega_1^2\omega_2^2 + (\omega_1^2 - 4\tau^2\omega_1^2\omega_2^2 + \omega_2^2)$$
$$(k-\tau\omega_1^2-\tau\omega_2^2) \Big] + Ba\omega_1 \Big[\omega_2(k-\tau\omega_1^2-\tau\omega_2^2)^2 - \omega_1^2\omega_2$$
$$- 4\tau^2\omega_1^2\omega_2^3 + \omega_2^3 \Big] \qquad \text{(A-84)}$$

$$\Delta G = -Aa\omega_1 \Big[\omega_2(k-\tau\omega_1^2-\tau\omega_2^2)^2 + 4\tau^2\omega_1^2\omega_2^3 - \omega_1^2\omega_2$$
$$+ 2\tau\omega_1^2\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) + 2\tau\omega_1^2\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_2^3 \Big]$$
$$+ Ba\omega_1 \Big[(k-\tau\omega_1^2-\tau\omega_2^2)^3 + 2\tau\omega_1^2\omega_2^2 - 2\tau\omega_1^2\omega_2^2 + \omega_1^2(k-\tau\omega_1^2-\tau\omega_2^2)$$
$$+ 4\tau^2\omega_1^2\omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) + \omega_2^2(k-\tau\omega_1^2-\tau\omega_2^2) \Big] \qquad \text{(A-85)}$$

$$\Delta G = Aa\omega_1\left[\omega_1^2\omega_2 - \omega_2(k-\tau\omega_1^2-\tau\omega_2^2)^2 - 4\tau^2\omega_1^2\omega_2^3 - 4\tau\omega_1^2\omega_2(k-\tau\omega_1^2-\tau\omega_2^2) - \omega_2^3\right] + Ba\omega_1\left[(k-\tau\omega_1^2-\tau\omega_2^2)^3 + (\omega_1^2+4\tau^2\omega_1^2\omega_2^2+\omega_2^2)(k-\tau\omega_1^2-\tau\omega_2^2)\right] \quad \text{(A-85)}$$

$$(k - a\omega_1\cos\omega_1 t)\,x_4 + \dot{x}_4 + \tau\ddot{x}_4 = (a\omega_1\cos\omega_1 t)\,x_3 \quad \text{(A-66a)}$$

Let

$$x_4 = x_5 + x_6 \quad \text{(A-86)}$$

$$x_3 = D\sin\omega_1 t\sin\omega_2 t + E\sin\omega_1 t\cos\omega_2 t + F\cos\omega_1 t\sin\omega_2 t + G\cos\omega_1 t\cos\omega_2 t \quad \text{(A-75)}$$

where: $$k x_5 + \dot{x}_5 + \tau\ddot{x}_5 = (a\omega_1\cos\omega_1 t)\,x_3 \quad \text{(A-63)}$$

Substitute (A-86) and (A-75) into (A-66a).

$$(k - a\omega_1\cos\omega_1 t)(x_5+x_6) + (\dot{x}_5+\dot{x}_6) + \tau(\ddot{x}_5+\ddot{x}_6) = Da\omega_1\sin\omega_1 t\cos\omega_1 t\sin\omega_2 t + Ea\omega_1\sin\omega_1 t\cos\omega_1 t\cos\omega_2 t + Fa\omega_1\cos^2\omega_1 t\sin\omega_2 t + Ga\omega_1\cos^2\omega_1 t\cos\omega_2 t \quad \text{(A-87)}$$

Try a solution for x_5 where:

$$kx_5 + \dot{x}_5 + \tau\ddot{x}_5 = \frac{Da\omega_1}{2}\sin 2\omega_1 t\sin\omega_2 t + \frac{Ea\omega_1}{2}\sin 2\omega_1 t\cos\omega_2 t + \left[\frac{Fa\omega_1}{2}\sin\omega_2 t + \frac{Fa\omega_1}{2}\cos 2\omega_1 t\sin\omega_2 t\right] + \left[\frac{Ga\omega_1}{2}\cos\omega_2 t + \frac{Ga\omega_1}{2}\cos 2\omega_1 t\cos\omega_2 t\right] \quad \text{(A-88)}$$

Let

$$x_5 = H_1\sin 2\omega_1 t\sin\omega_2 t + I\sin 2\omega_1 t\cos\omega_2 t + J\cos 2\omega_1 t\sin\omega_2 t + L\cos 2\omega_1 t\cos\omega_2 t + M\sin\omega_2 t + N\cos\omega_2 t \quad \text{(A-89)}$$

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x_5 could be written in an equivalent form with certain terms containing $(2\omega_1 t - \omega_2 t)$ by use of trigonometry relations. Thus a time dependent drift term is seen to appear when the harmonic situation $2\omega_1 = \omega_2$ exists.

$$\begin{aligned}\dot{x}_5 = {} & 2H_1\,\omega_1 \cos 2\omega_1 t \sin \omega_2 t + H_1\,\omega_2 \sin 2\omega_1 t \cos \omega_2 t \\ & + 2I\,\omega_1 \cos 2\omega_1 t \cos \omega_2 t - I\,\omega_2 \sin 2\omega_1 t \sin \omega_2 t \\ & - 2J\,\omega_1 \sin 2\omega_1 t \sin \omega_2 t + J\,\omega_2 \cos 2\omega_1 t \cos \omega_2 t \\ & - 2L\,\omega_1 \sin 2\omega_1 t \cos \omega_2 t - L\,\omega_2 \cos 2\omega_1 t \sin \omega_2 t \\ & + M\,\omega_2 \cos \omega_2 t - N\,\omega_2 \sin \omega_2 t\end{aligned} \qquad \text{(A-89a)}$$

Combining terms of x_5, one obtains:

$$\begin{aligned}\dot{x}_5 = {} & (2H_1\omega_1 - L\omega_2) \cos 2\omega_1 t \sin \omega_2 t + (H_1\omega_2 - 2L\omega_1) \sin 2\omega_1 t \cos \omega_2 t \\ & + (2I\omega_1 + J\omega_2) \cos 2\omega_1 t \cos \omega_2 t - (I\omega_2 + 2J\omega_1) \sin 2\omega_1 t \sin \omega_2 t \\ & + (M\omega_2) \cos \omega_2 t - (N\omega_2) \sin \omega_2 t\end{aligned} \qquad \text{(A-89a)}$$

$$\begin{aligned}\ddot{x}_5 = {} & (2H_1\omega_1 - L\omega_2)(-2\omega_1) \sin 2\omega_1 t \sin \omega_2 t + (2H_1\omega_1 - L\omega_2)(\omega_2) \cos 2\omega_1 t \cos \omega_2 t \\ & + (H_1\omega_2 - 2L\omega_1)(2\omega_1) \cos 2\omega_1 t \cos \omega_2 t + (H_1\omega_2 - 2L\omega_1)(-\omega_2) \sin 2\omega_1 t \sin \omega_2 t \\ & + (2I\omega_1 + J\omega_2)(-2\omega_1) \sin 2\omega_1 t \cos \omega_2 t + (2I\omega_1 + J\omega_2)(-\omega_2) \cos 2\omega_1 t \sin \omega_2 t \\ & - (I\omega_2 + 2J\omega_1)(2\omega_1) \cos 2\omega_1 t \sin \omega_2 t - (I\omega_2 + 2J\omega_1)(\omega_2) \sin 2\omega_1 t \cos \omega_2 t \\ & - (M\,\omega_2^2) \sin \omega_2 t - (N\,\omega_2^2) \cos \omega_2 t\end{aligned} \qquad \text{(A-89b)}$$

Combining terms of x_5, one obtains:

$$\begin{aligned}\ddot{x}_5 = {} & (2L\omega_1\omega_2 - 4H_1\,\omega_1^2 - H_1\,\omega_2^2 + 2L\omega_1\omega_2) \sin 2\omega_1 t \sin \omega_2 t \\ & + (2H_1\omega_1\omega_2 - L\,\omega_2^2 + 2H_1\omega_1\omega_2 - 4L\,\omega_1^2) \cos 2\omega_1 t \cos \omega_2 t \\ & + (-4I\,\omega_1^2 - 2J\omega_1\omega_2 - I\,\omega_2^2 - 2J\omega_1\omega_2) \sin 2\omega_1 t \cos \omega_2 t \\ & + (-2I\omega_1\omega_2 - J\,\omega_2^2 - 2I\omega_1\omega_2 - 4J\,\omega_1^2) \cos 2\omega_1 t \sin \omega_2 t \\ & + (-M\,\omega_2^2) \sin \omega_2 t + (-N\,\omega_2^2) \cos \omega_2 t\end{aligned} \qquad \text{(A-89b)}$$

Now substitute (A-89), (A-89a), and (A-89b) into (A-88) and solve for coefficients H_1, I, J, L, M, and N.

$$\Big[kH_1 \sin 2\omega_1 t \sin \omega_2 t + kI \sin 2\omega_1 t \cos \omega_2 t + kJ \cos 2\omega_1 t \sin \omega_2 t$$
$$+ kL \cos 2\omega_1 t \cos \omega_2 t + kM \sin \omega_2 t + kN \cos \omega_2 t\Big] +$$
$$\Big[(2H_1\omega_1 - L\omega_2) \cos 2\omega_1 t \sin \omega_2 t + (H_1\omega_2 - 2L\omega_1) \sin 2\omega_1 t \cos \omega_2 t$$
$$+ (2I\,\omega_1 + J\omega_2) \cos 2\omega_1 t \cos \omega_2 t - (I\,\omega_2 + 2J\,\omega_1) \sin 2\omega_1 t \sin \omega_2 t$$
$$+ (M\,\omega_2) \cos \omega_2 t - (N\,\omega_2) \sin \omega_2 t\Big] +$$
$$\Big[\tau\,(2L\,\omega_1\omega_2 - 4H_1\,\omega_1^2 - H_1\,\omega_2^2 + 2L\,\omega_1\omega_2) \sin 2\omega_1 t \sin \omega_2 t$$
$$+ \tau(2H_1\,\omega_1\omega_2 - L\omega_2^2 + 2H_1\,\omega_1\omega_2 - 4L\,\omega_1^2) \cos 2\omega_1 t \cos \omega_2 t$$
$$+ \tau(-4I\,\omega_1^2 - 2J\,\omega_1\omega_2 - I\,\omega_2^2 - 2J\,\omega_1\omega_2) \sin 2\omega_1 t \cos \omega_2 t$$
$$+ \tau(-2I\,\omega_1\omega_2 - J\,\omega_2^2 - 2I\,\omega_1\omega_2 - 4J\,\omega_1^2) \cos 2\omega_1 t \sin \omega_2 t$$
$$+ \tau\,(-M\,\omega_2^2) \sin \omega_2 t + \tau(-N\,\omega_2^2) \cos \omega_2 t\Big] =$$
$$\Bigg[\left(\frac{Da\omega_1}{2}\right) \sin 2\omega_1 t \sin \omega_2 t + \left(\frac{Ea\omega_1}{2}\right) \sin 2\omega_1 t \cos \omega_2 t$$
$$+ \left(\frac{Fa\omega_1}{2}\right) \sin \omega_2 t + \left(\frac{Fa\omega_1}{2}\right) \cos 2\omega_1 t \sin \omega_2 t$$
$$+ \left(\frac{Ga\omega_1}{2}\right) \cos \omega_2 t + \left(\frac{Ga\omega_1}{2}\right) \cos 2\omega_1 t \cos \omega_2 t\Bigg] \qquad \text{(A-90)}$$

Now equate like terms of the various trigonometric functions and solve for H_1, I, J, L, M, and N. (Note: A, B, C, D, E, and F are known)

<u>$\sin 2\omega_1 t \sin \omega_2 t$</u>

$$kH_1 - (I\omega_2 + 2J\,\omega_1) + \tau(4L\omega_1\omega_2 - 4H_1\,\omega_1^2 - H_1\,\omega_2^2 = \frac{Da\omega_1}{2}$$

$$H_1\,(k - 4\,\tau\omega_1^2 - \tau\,\omega_2^2) + (-\omega_2)\,I + (-2\omega_1)\,J + (4\,\tau\,\omega_1\omega_2)\,L = \frac{Da\omega_1}{2} \qquad \text{(A-91)}$$

$\underline{\sin 2\omega_1 t \cos \omega_2 t}$

$$kI + (H_1 \omega_2 - 2L\omega_1) + \tau(-4I\omega_1^2 - 4J\omega_1\omega_2 - I\omega_2^2) = \frac{Ea\omega_1}{2}$$

$$H_1(\omega_2) + I(k - 4\tau\omega_1^2 - \tau\omega_2^2) + J(-4\tau\omega_1\omega_2) + L(-2\omega_1) = \frac{Ea\omega_1}{2} \tag{A-92}$$

$\underline{\cos 2\omega_1 t \sin \omega_2 t}$

$$kJ + (2H\omega_1 - L\omega_2) + \tau(-4I\omega_1\omega_2 - J\omega_2^2 - 4J\omega_1^2) = \frac{Fa\omega_1}{2}$$

$$H(2\omega_1) + I(-4\tau\omega_1\omega_2) + J(k - \tau\omega_2^2 - 4\tau\omega_1^2) + L(-\omega_2) = \frac{Fa\omega_1}{2} \tag{A-93}$$

$\underline{\cos 2\omega_1 t \cos \omega_2 t}$

$$kL + (2I\omega_1 + J\omega_2) + \tau(2H\omega_1\omega_2 - L\omega_2^2 + 2H\omega_1\omega_2 - 4L\omega_1^2) = \frac{Ga\omega_1}{2}$$

$$H(4\tau\omega_1\omega_2) + I(2\omega_1) + J(\omega_2) + L(k - \tau\omega_2^2 - 4\tau\omega_1^2) = \frac{Ga\omega_1}{2} \tag{A-94}$$

$\underline{\cos \omega_2 t}$

$$M\omega_2 - \tau N\omega_2^2 = \frac{Ga\omega_1}{2} \qquad M(\omega_2) + N(-\tau\omega_2^2) = \frac{Ga\omega_1}{2} \tag{A-95}$$

$\underline{\sin \omega_2 t}$

$$-N\omega_2 - \tau M\omega_2^2 = \frac{Fa\omega_1}{2} \qquad M(-\tau\omega_2^2) + N(-\omega_2) = \frac{Fa\omega_1}{2} \tag{A-96}$$

Solve (A-95) and (A-96) for M and N.

$$M = \tau N\omega_2 + \frac{Ga\omega_1}{2\omega_2} \tag{A-95a}$$

Substitute (A-95a) into (A-96).

$$- N\omega_2 - \tau\omega_2^2 \left[\tau N\omega_2 + \frac{Ga\omega_1}{2\omega_2} \right] = \frac{Fa\omega_1}{2}$$

$$- N\omega_2 - N\tau^2\omega_2^3 - \frac{G\tau a\omega_1\omega_2}{2} = \frac{Fa\omega_1}{2}$$

$$N(-\omega_2 - \tau^2\omega_2^3) = \frac{Fa\omega_1 + \tau Ga\omega_1\omega_2}{2}$$

$$N = \frac{a\omega_1(F + \tau G\omega_2)}{-2\omega_2(1 + \tau^2\omega_2^2)}$$

$$\boxed{N = \frac{-a\omega_1(F + \tau G\omega_2)}{2\omega_2(1 + \tau^2\omega_2^2)}} \qquad \text{(A-97)}$$

$$N = -\tau M\omega_2 - \frac{Fa\omega_1}{2\omega_2} \qquad \text{(A-96a)}$$

Substitute (A-96a) into (A-95).

$$M\omega_2 - \tau\omega_2^2 \left[-\tau M\omega_2 - \frac{Fa\omega_1}{2\omega_2} \right] = \frac{Ga\omega_1}{2} \qquad \text{(A-95)}$$

$$M\omega_2 + \tau^2 M\omega_2^3 + \frac{\tau Fa\omega_1\omega_2}{2} = \frac{Ga\omega_1}{2}$$

$$M(\omega_2 + \tau^2\omega_2^3) = \frac{Ga\omega_1 - \tau Fa\omega_1\omega_2}{2}$$

$$\boxed{M = \frac{a\omega_1(G - \tau F\omega_2)}{2\omega_2(1 + \tau^2\omega_2^2)}} \qquad \text{(A-98)}$$

$$(k - 4\tau\omega_1^2 - \tau\omega_2^2) H_1 + (-\omega_2) I + (-2\omega_1) J + (4\tau\omega_1\omega_2) L = \frac{Da\omega_1}{2} \tag{A-91}$$

$$(\omega_2) H_1 + (k - 4\tau\omega_1^2 - \tau\omega_2^2) I + (-4\tau\omega_1\omega_2) J + (-2\omega_1) L = \frac{Ea\omega_1}{2} \tag{A-92}$$

$$(2\omega_1) H_1 + (-4\tau\omega_1\omega_2) I + (k - \tau\omega_2^2 - 4\tau\omega_1^2) J + (-\omega_2) L = \frac{Fa\omega_1}{2} \tag{A-93}$$

$$(4\tau\omega_1\omega_2) H_1 + (2\omega_1) I + (\omega_2) J + (k - \tau\omega_2^2 - 4\tau\omega_1^2) L = \frac{Ga\omega_1}{2} \tag{A-94}$$

Solve for H_1, I, J, and L by determinants.

(A-99)

$$H_1 = \frac{\begin{vmatrix} \frac{Da\omega_1}{2} & (-\omega_2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ \frac{Ea\omega_1}{2} & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ \frac{Fa\omega_1}{2} & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\ \frac{Ga\omega_1}{2} & (2\omega_1) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega 2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}$$

(A-100)

Let the denominator of the determinant be called λ or (A-100).

$$I = \frac{\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & \frac{Da\omega_1}{2} & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ (\omega_2) & \frac{Ea\omega_1}{2} & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (2\omega_1) & \frac{Fa\omega_1}{2} & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\ (4\tau\omega_1\omega_2) & \frac{Ga\omega_1}{2} & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}}{\lambda} \tag{A-101}$$

$$J = \frac{\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & \frac{Da\omega_1}{2} & (4\tau\omega_1\omega_2) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & \frac{Ea\omega_1}{2} & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & \frac{Fa\omega_1}{2} & (-\omega_2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & \frac{Ga\omega_1}{2} & (k - 4\tau\omega_1^2 - \omega_2^2) \end{vmatrix}}{\lambda} \tag{A-102}$$

$$L = \frac{\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (-2\omega_1) & \frac{Da\omega_1}{2} \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & \frac{Ea\omega_1}{2} \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & \frac{Fa\omega_1}{2} \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) & \frac{Ga\omega_1}{2} \end{vmatrix}}{\lambda} \tag{A-103}$$

$$\lambda = (k-4\tau\omega_1^2-\tau\omega_2^2)\begin{vmatrix} (k-4\tau\omega_1^2-\tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (-4\tau\omega_1\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) & (-\omega_2) \\ (2\omega_1) & (\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) \end{vmatrix}$$

$$-(-\omega_2)\begin{vmatrix} (\omega_2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (2\omega_1) & (k-4\tau\omega_1^2-\tau\omega_2^2) & (-\omega_2) \\ (4\tau\omega_1\omega_2) & (\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) \end{vmatrix}$$

$$+(-2\omega_1)\begin{vmatrix} (\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (-\omega_2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (k-4\tau\omega_1^2-\tau\omega_2^2) \end{vmatrix}$$

$$-(4\tau\omega_1\omega_2)\begin{vmatrix} (\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) & (-4\tau\omega_1\omega_2) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k-4\tau\omega_1^2-\tau\omega_2^2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) \end{vmatrix} \tag{A-100}$$

$$\begin{aligned}
\lambda = {} & (k-4\tau\omega_1^2-\tau\omega_2^2)\Big[(k-4\tau\omega_1^2-\tau\omega_2^2)^3 + (8\tau\omega_1^2\omega_2^2) + (8\tau\omega_1^2\omega_2^2) \\
& -(-4\omega_1^2)(k-4\tau\omega_1^2-\tau\omega_2^2) - (16\tau^2\omega_1^2\omega_2^2)(k-4\tau\omega_1^2-\tau\omega_2^2) \\
& -(-\omega_2^2)(k-4\tau\omega_1^2-\tau\omega_2^2)\Big] \\
& + (\omega_2)\Big[(\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2)^2 + (16\tau^2\omega_1^2\omega_2^3) - (4\omega_1^2\omega_2) \\
& - (-8\tau\omega_1^2\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2) - (-8\tau\omega_1^2\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2) + (\omega_2^3)\Big] \\
& + (-2\omega_1)\Big[(-4\tau\omega_1\omega_2^2)(k-4\tau\omega_1^2-\tau\omega_2^2) + (-4\tau\omega_1\omega_2^2)(k-4\tau\omega_1^2-\tau\omega_2^2) \\
& + (-8\omega_1^3) - (32\tau^2\omega_1^3\omega_2^2) - (2\omega_1)(k-4\tau\omega_1^2-\tau\omega_2^2)^2 - (-2\omega_1\omega_2^2)\Big] \\
& + (-4\tau\omega_1\omega_2)\Big[(-4\tau\omega_1\omega_2^3) + (4\tau\omega_1\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2)^2 + (-16\tau\omega_1^3\omega_2) \\
& - (64\tau^3\omega_1^3\omega_2^3) - (2\omega_1\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2) - (2\omega_1\omega_2)(k-4\tau\omega_1^2-\tau\omega_2^2)\Big]
\end{aligned} \tag{A-100}$$

$$\lambda = \Big[(k - \tau\omega_1^2 - \tau\omega_2^2)^4 + (8\omega_1^2 - 32\,\tau^2\omega_1^2\omega_2^2 + 2\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2$$

$$+ (64\,\tau\omega_1^2\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (\omega_2^4 + 16\omega_1^4 + 32\,\tau^2\omega_1^2\omega_2^4$$

$$+ 128\,\tau^2\omega_1^4\omega_2^2 + 256\,\omega_1^4\omega_2^4\,\tau^4 - 8\,\omega_1^2\omega_2^2 \Big] \qquad \text{(A-100)}$$

$$\lambda H_1 = \frac{Da\omega_1}{2} \begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\ (2\omega_1) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$- \frac{Ea\omega_1}{2} \begin{vmatrix} (-\omega_2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\ (2\omega_1) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$+ \frac{Fa\omega_1}{2} \begin{vmatrix} (-\omega_2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (2\omega_1) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

$$- \frac{Ga\omega_1}{2} \begin{vmatrix} (-\omega_2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\ (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\ (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \end{vmatrix}$$

(A-99)

$$
\begin{aligned}
\lambda H_1 = \left(\frac{Da\omega_1}{2}\right) \Big[& (k - 4\tau\omega_1^2 - \tau\omega_2^2)^3 + (8\tau\omega_1^2\omega_2^2) + (8\tau\omega_1^2\omega_2^2) \\
& + (4\omega_1^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) - (16\tau^2\omega_1^2\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) \\
& + (\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) \Big] \\
\left(\frac{Ea\omega_1}{2}\right) \Big[& (\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 - (4\omega_1^2\omega_2) + (16\tau^2\omega_1^2\omega_2^3) \\
& + (8\tau\omega_1^2\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (8\tau\omega_1^2\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (\omega_2^3) \Big] \\
\left(\frac{Fa\omega_1}{2}\right) \Big[& (4\tau\omega_1\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (8\omega_1^3) \\
& + (4\tau\omega_1\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (32\tau^2\omega_1^3\omega_2^2) + (2\omega_1)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 \\
& - (2\omega_1\omega_2^2) \Big] \\
\left(\frac{Ga\omega_1}{2}\right) \Big[& (4\tau\omega_1\omega_2^3) + (16\tau\omega_1^3\omega_2) - (4\tau\omega_1\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 \\
& + (64\tau^3\omega_1^3\omega_2^3) + (4\omega_1\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) \Big]
\end{aligned}
\tag{A-99}
$$

$$
\begin{aligned}
\lambda H_1 = &\left(\frac{Da\omega_1}{2}\right)\Big[(k - 4\tau\omega_1^2 - \tau\omega_2^2)^3 + (16\tau\omega_1^2\omega_2^2) + \\
&\quad + (4\omega_1^2 - 16\tau^2\omega_1^2\omega_2^2 + \omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)\Big] + \\
&\left(\frac{Ea\omega_1}{2}\right)\Big[(\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 - (4\omega_1^2\omega_2) + (16\tau^2\omega_1^2\omega_2^3) + \\
&\quad + (16\tau\omega_1^2\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (\omega_2^3)\Big] + \\
&\left(\frac{Fa\omega_1}{2}\right)\Big[(2\omega_1)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 - (2\omega_1\omega_2^2) + 32\tau^2\omega_1^3\omega_2^2 + \\
&\quad + (8\tau\omega_1\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (8\omega_1^3)\Big] + \\
&\left(\frac{Ga\omega_1}{2}\right)\Big[(-4\tau\omega_1\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 + 4\tau\omega_1\omega_2^3 + \\
&\quad + 64\tau^3\omega_1^3\omega_2^3 + (4\tau\omega_1\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + \\
&\quad + 16\tau\omega_1^3\omega_2\Big]
\end{aligned}
\tag{A-99}
$$

$$
\lambda I = -\left(\frac{Da\omega_1}{2}\right)
\begin{vmatrix}
(\omega_2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\
(2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\
(4\tau\omega_1\omega_2) & (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2)
\end{vmatrix}
$$

$$
+\left(\frac{Ea\omega_1}{2}\right)
\begin{vmatrix}
(k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\
(2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) \\
(4\tau\omega_1\omega_2) & (\omega_2) & (k - 4\tau\,\omega_1^2 - \tau\omega_2^2)
\end{vmatrix}
$$

$$
-\left(\frac{Fa\omega_1}{2}\right)
\begin{vmatrix}
(k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\
(\omega_2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\
(4\tau\omega_1\omega_2) & (\omega_2) & (k - \tau\omega_1^2 - \tau\omega_2^2)
\end{vmatrix}
$$

$$
+\left(\frac{Ga\omega_1}{2}\right)
\begin{vmatrix}
(k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) & (4\tau\omega_1\omega_2) \\
(\omega_2) & (-4\tau\omega_1\omega_2) & (-2\omega_1) \\
(2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2)
\end{vmatrix}
\tag{A-101}
$$

$$
\begin{aligned}
\lambda I = & -\left(\frac{Da\omega_1}{2}\right)\Bigg[(\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 + (16\tau^2\omega_1^2\omega_2^3) - (4\omega_1^2\omega_2) + \\
& + (8\tau\omega_1^2\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (8\tau\omega_1^2\omega_2)(k-4\tau\omega_1^2 - \tau\omega_2^2) + \\
& + (\omega_2^3)\Bigg] + \\
& + \left(\frac{Ea\omega_1}{2}\right)\Bigg[(k - 4\tau\omega_1^2 - \tau\omega_2^2)^3 + (8\tau\omega_1^2\omega_2^2) + (8\tau\omega_1^2\omega_2^2) + \\
& - (16\tau^2\omega_1^2\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (4\omega_1^2)(k-4\tau\omega_1^2 - \tau\omega_2^2) + \\
& + (\omega_2^2)(k-4\tau\omega_1^2 - \tau\omega_2^2)\Bigg] + \\
& - \left(\frac{Fa\omega_1}{2}\right)\Bigg[(-4\tau\omega_1\omega_2)(k - 4\tau\omega_1^2 - \tau\omega_2^2)^2 + (16\tau\omega_1^3\omega_2) + \\
& + (4\tau\omega_1\omega_2^3) - (64\tau^3\omega_1^3\omega_2^3) + (2\omega_1\omega_2)(k-4\tau\omega_1^2 - \tau\omega_2^2) + \\
& + (2\omega_1\omega_2)(k-4\tau\omega_1^2 - \tau\omega_2^2)\Bigg] + \\
& + \left(\frac{Ga\omega_1}{2}\right)\Bigg[(4\tau\omega_1\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (8\omega_1^3) + \\
& + (4\tau\omega_1\omega_2^2)(k - 4\tau\omega_1^2 - \tau\omega_2^2) + (32\tau^2\omega_1^3\omega_2^2) - (2\omega_1\omega_2^2) + \\
& + (2\omega_1)(k-4\tau\omega_1^2 - \tau\omega_2^2)^2\Bigg]
\end{aligned}
\tag{A-101}
$$

$$\lambda \mathbf{J} = \left(\frac{\mathbf{Da}\omega_1}{2}\right) \begin{vmatrix} (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (-\omega_2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix} +$$

$$+ \left(\frac{-\mathbf{Ea}\omega_1}{2}\right) \begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (4\tau\omega_1\omega_2) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (-\omega_2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix} +$$

$$+ \left(\frac{\mathbf{Fa}\omega_1}{2}\right) \begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (4\tau\omega_1\omega_2) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix} +$$

$$+ \left(\frac{-\mathbf{Ga}\omega_1}{2}\right) \begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (4\tau\omega_1\omega_2) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (-\omega_2) \end{vmatrix}$$

(A-102)

$$\lambda L = \left(-\frac{Da\omega_1}{2}\right)\begin{vmatrix} (\omega_2) & (k - \tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) \end{vmatrix} +$$

$$+ \left(\frac{Ea\omega_1}{2}\right)\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (-2\omega_1) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) \end{vmatrix} +$$

$$+ \left(\frac{Fa\omega_1}{2}\right)\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (-2\omega_1) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) \\ (4\tau\omega_1\omega_2) & (2\omega_1) & (\omega_2) \end{vmatrix} +$$

$$- \left(\frac{Ga\omega_1}{2}\right)\begin{vmatrix} (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-\omega_2) & (-2\omega_1) \\ (\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) & (-4\tau\omega_1\omega_2) \\ (2\omega_1) & (-4\tau\omega_1\omega_2) & (k - 4\tau\omega_1^2 - \tau\omega_2^2) \end{vmatrix}$$

(A-103)

λJ and λL are left in the form of a third order determinant. Further determinant breakdown will yield terms similar to those associated with λH and λI.

The determinant form does permit the detection of the common factor "a" in all the coefficients. As in the case for zero elastic restraint, terms beyond x_5 may be neglected if the order of magnitude of "a" is much smaller than unity. Each successive term in the series of x is smaller than its predecessor in accordance with the value of "a".

$$x = x_1 + x_3 + x_5 + \ldots + x_{2m-1} + x_{2m} \qquad \text{(A-3)}$$

$$x_1 = A \sin \omega_2 t + B \cos \omega_2 t \qquad \text{(A-67)}$$

$$\begin{aligned} x_3 = {} & D \sin \omega_1 t \sin \omega_2 t + E \sin \omega_1 t \cos \omega_2 t \\ & + F \cos \omega_1 t \sin \omega_2 t + G \cos \omega_1 t \cos \omega_2 t \end{aligned} \qquad \text{(A-75)}$$

$$\begin{aligned} x_5 = {} & H_1 \sin 2\omega_1 t \sin \omega_2 t + I \sin 2\omega_1 t \cos \omega_2 t \\ & + J \cos 2\omega_1 t \sin \omega_2 t + L \cos 2\omega_1 t \cos \omega_2 t \\ & + M \sin \omega_2 t + N \cos \omega_2 t \end{aligned} \qquad \text{(A-89)}$$

Drift terms may be shown to exist by applying trigonometry relations to x_3 and x_5. Equal input frequencies and harmonics will cause time independent terms of constant angular displacement to appear but there are no time dependent terms as in the case of zero elastic restraint.

$$x_3 = \frac{D}{2}\left[\cos(\omega_1 t - \omega_2 t) - \cos(\omega_1 t + \omega_2 t)\right]$$
$$+ \frac{E}{2}\left[\sin(\omega_1 t + \omega_2 t) + \sin(\omega_1 t - \omega_2 t)\right]$$
$$+ \frac{F}{2}\left[\sin(\omega_1 t + \omega_2 t) - \sin(\omega_1 t - \omega_2 t)\right]$$
$$+ \frac{G}{2}\left[\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t)\right] \qquad \text{(A-75a)}$$

$$x_5 = \frac{H}{2}\left[\cos(2\omega_1 t - \omega_2 t) - \cos(2\omega_1 t + \omega_2 t)\right]$$
$$+ \frac{I}{2}\left[\sin(2\omega_1 t + \omega_2 t) + \sin(2\omega_1 t - \omega_2 t)\right]$$
$$+ \frac{J}{2}\left[\sin(2\omega_1 t + \omega_2 t) - \sin(2\omega_1 t - \omega_2 t)\right]$$
$$+ \frac{L}{2}\left[\cos(2\omega_1 t + \omega_2 t) + \cos(2\omega_1 t - \omega_2 t)\right]$$
$$+ M \sin \omega_2 t + N \cos \omega_2 t \qquad \text{(A-89a)}$$

x_1 terms are pure sinusoids and the time average value at the end of complete cycles is zero. x_3 terms are sinusoids which do not display time dependent characteristics if $\omega_1 = \omega_2$. Constant displacements such as D/2 and G/2 do appear when $\omega_1 = \omega_2$. Likewise in x_5 terms, time dependent terms are lacking when $2\omega_1 = \omega_2$ but constant angular displacements such as H/2 and L/2 are present. This phenomena can be easily explained through the coefficients attached to the sinusoids. The denominators of such coefficients become zero for $\omega_1 = \omega_2$ and $2\omega_1 = \omega_2$ in the case for zero elastic restraint. However, a finite elastic restraint keeps a real value attached to the coefficients and thus indeterminate forms do not appear when $\sin(\omega_1 t - \omega_2 t)$ and $\sin(2\omega_1 t - \omega_2 t)$ are zero.

[illegible]

This analysis is verified by Ref. 10 for the case where $\omega_1 = \omega_2$ and k is finite.

One other possibility is seen to exist. It concerns Δ and λ, the denominators associated with the determinant forms of the coefficients. If Δ and λ can be made equal to zero by some combination of ω_1 and ω_2, drifts could possibly occur. Attempts to solve Δ for ω_1 to give $\Delta = 0$, lead to an equation in ω_1 of the eighth order. Similar difficulty appears when λ is set equal to zero and an attempt is made to solve for ω_1. With such values of ω_1 and ω_2 which satisfy $\Delta = 0$ or $\lambda = 0$, L' Hospital's Rule could be applied and the possibility of a drift rate could be investigated. This appears as a field for more complete investigation.

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